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MOMENTS OF FIRST-PASSAGE DISTRIBUTIONS IN SLAB GEOMETRY

by

P. J. Brockwell and J. M. Landwehr

ANL-7519
Mathematics and
Computers

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

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Applied Mathematics Division

December 1968

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I. INTRODUCTION

There exists a variety of physical processes having a Markovian character in which a particle suffers a succession of independent collisions with atoms of the medium through which it travels. Examples are the following: the diffuse scattering of light, where the particle is a photon; the diffusion of neutrons; the multiple scattering of charged particles. In connection with such processes, one is often interested in the probability distribution of the state variables of the particles (position, velocity, wavelength, spin, etc.) when they make first passages through some surface in space (regardless of the times of emergence). A general stochastic theory of such first-passage distributions has been developed by Moyal,[10,11] and Brockwell.[3] For a comprehensive account of the theory of branching processes as applied to neutron transport theory, we refer the reader to Mullikin.[13]

The following specific transport model will be dealt with in this report. We shall be concerned with a population of particles (for example, neutrons), the state of an individual particle being represented by a vector $\underset{\sim}{y} = (\underset{\sim}{x}, \underset{\sim}{\mu})$, where $\underset{\sim}{x}$ denotes position and $\underset{\sim}{\mu}$ denotes direction of motion. (Particle speed is assumed to be essentially constant—this is the classical one-group approximation.) We denote by Y the set of all possible individual states and by $\mathcal{B}(Y)$ the Borel field of subsets of Y. The population state space $\mathcal{Y}$ is then defined by

$$\mathcal{Y} = \bigcup_{n=0}^{\infty} Y^n,$$

where Y^n, $n \geq 1$, denotes the n-fold Cartesian product of Y with itself, and Y^0 denotes the state in which the population is empty. Measurable subsets of $\mathcal{Y}$ are defined to be those belonging to the σ-field $\mathcal{B}(\mathcal{Y})$ generated by the product σ-fields $\mathcal{B}(Y^n)$, $n = 0, 1, 2, \ldots$. A typical element of $\mathcal{Y}$ corresponding to a population of size n will be denoted by $\underset{\sim}{y}^{(n)}$. (We imagine the particles to be labeled in some way so that $\underset{\sim}{y}^{(n)}$ is an <u>ordered</u> n-tuple.) For a detailed account of the theory of stochastic population processes, see Moyal.[9]

We suppose that the behavior of the population of particles is governed by the following functions (which we assume known from quantum-mechanical or other considerations):

1. The total cross section or inverse mean-free-path, $\lambda(\underset{\sim}{x})$

Given a particle with state $\underset{\sim}{y} = (\underset{\sim}{x}, \underset{\sim}{\mu})$, the probability that it experiences a collision in traveling a small distance $d\ell$ is $\lambda(\underset{\sim}{x})d\ell + o(d\ell)$. The probability of more than one collision in $d\ell$ is $o(d\ell)$.

2. The collision outcome distribution, $\{\phi_n\}$

The distribution $\{\phi_n\}$ is a conditional probability measure on $\mathcal{B}(\mathcal{Y}) \times Y$. More specifically, $\phi_n(A|\underset{\sim}{y})$ is the probability, given that a particle with state $\underset{\sim}{y}$ experiences a collision, that exactly n particles are produced with states $\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_n$ such that $\underset{\sim}{y}^{(n)} = (\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_n) \in A$. Note that the colliding particle itself (if it survives the collision) is included among the particles produced. For photon scattering, where only absorption or scattering are possible, $\phi_n = 0$ for $n \geq 2$. In general,

$$\sum_{n=1}^{\infty} \phi_n(Y^n|\underset{\sim}{y}) = 1 - \phi_0(\underset{\sim}{y}),$$

where $\phi_0(\underset{\sim}{y})$ is the probability that the colliding particle is absorbed.

In addition to assuming that λ and $\{\phi_n\}$ are known, we assume that between collisions each particle experiences a change in position only (i.e., $\underset{\sim}{\mu}$ is constant). Moreover, we suppose that there is no interaction between the particles themselves, only between the particles and the atoms of the medium through which they move.

The problems we shall discuss here fall into the general class of first-passage problems (see Moyal[10]). Given a single initial particle in a slab of thickness t (infinite in two directions), we specify its position by a single coordinate $x\,(0 \leq x \leq t)$ and its direction of motion by the cosine μ of the angle it makes with the positive x-axis (normal to the slab faces). The problem can be stated as follows: Given a single initial particle with state $\underset{\sim}{y} = (x, \mu)$, what is the probability distribution of the resulting population of particles making first passages from the slab (the state of each particle in this population being its state as it emerges from the slab)? If the exterior of the slab is vacuous (so that returns to the slab are impossible), then the first-passage population is identical with the total emergent population. We shall use the terms emergent population and first-passage population interchangeably.

For an arbitrary convex body with surface Σ, the number of particles making first passages from the body is a random number n, and the states of the emerging particles can be represented by an element $\underset{\sim}{y}^{(n)} = (\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_n)$ of the population state space $\mathcal{Y}$. It is useful to introduce the generating functional (conditional on a single initial particle with state $\underset{\sim}{y}$)

$$G(\xi \mid \underset{\sim}{y}) = E\xi(\underset{\sim}{y}_1)\xi(\underset{\sim}{y}_2) \ldots \xi(\underset{\sim}{y}_n)$$

$$\equiv P_0(\underset{\sim}{y}) + \sum_{n=1}^{\infty} \int \underset{Y^n}{\cdots} \int \xi(\underset{\sim}{y}_1) \ldots \xi(\underset{\sim}{y}_n) P_n(d\underset{\sim}{y}_1 \ldots d\underset{\sim}{y}_n \mid \underset{\sim}{y}),$$

where ξ is an arbitrary measurable function on Y such that

$$\sup_{\underset{\sim}{y} \in Y} |\xi(\underset{\sim}{y})| \leq 1$$

and $P_n(A \mid \underset{\sim}{y})$ is the probability, given a single initial particle with state $\underset{\sim}{y}$, that the state $\underset{\sim}{y}^{(n)}$ of the first-passage population satisfies $\underset{\sim}{y}^{(n)} \in A$. We then obtain the following stochastic form of the Boltzmann equation (see, for example, Moyal,[11] Pál,[14] Bell,[2] and Brockwell[4]):

$$[\underset{\sim}{\mu} \cdot \nabla - \lambda(\underset{\sim}{x})] G(\xi \mid \underset{\sim}{y}) = -\lambda(\underset{\sim}{x}) \left[\phi_0(\underset{\sim}{y}) + \sum_{n=1}^{\infty} \int \underset{Y^n}{\cdots} \int \phi_n(d\underset{\sim}{y}_1 \ldots d\underset{\sim}{y}_n \mid \underset{\sim}{y}) \prod_{i=1}^{n} G(\xi \mid \underset{\sim}{y}_i) \right], \tag{1}$$

with boundary conditions

$$G(\xi \mid \underset{\sim}{y}) = \xi(\underset{\sim}{y}) \text{ if } \underset{\sim}{x} \text{ is on the surface } \Sigma \text{ and } \underset{\sim}{\mu} \text{ is directed outwards,} \tag{2}$$

where $\underset{\sim}{y} = (\underset{\sim}{x}, \underset{\sim}{\mu})$.

The generating functional G uniquely determines (see Moyal[9]) a symmetric distribution on the measurable space $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$, which is just the probability distribution of the first-passage population. However, the equation for G is difficult to solve because of its nonlinearity. Instead, we shall study the associated moment distributions for which the Boltzmann equation is linear.

We shall assume that ϕ_n, $n = 1, 2, \ldots$, takes the form

$$\phi_n(d\underset{\sim}{y}_1 \ldots d\underset{\sim}{y}_n \mid \underset{\sim}{y}) = \Pi_n \left[\prod_{i=1}^{n} \delta(\underset{\sim}{x}_i - \underset{\sim}{x})\, \phi\,(d\underset{\sim}{\mu}_i \mid \underset{\sim}{\mu})\, d\underset{\sim}{x}_i \right], \tag{3}$$

where Π_n is the probability that exactly n particles result from the collision, δ is the three-dimensional Dirac delta function, and $\phi(d\underset{\sim}{\nu}|\underset{\sim}{\mu})$ is the probability that a particle produced by the collision has a direction of motion in the element of solid angle $d\underset{\sim}{\nu}$ about $\underset{\sim}{\nu}$. In subsequent sections, we will assume that $\phi(d\underset{\sim}{\nu}|\underset{\sim}{\mu})$ is of the form

$$\phi(d\underset{\sim}{\nu}|\underset{\sim}{\mu}) = \sigma(\theta)\, d\underset{\sim}{\nu}, \tag{4}$$

where θ is the angle between $\underset{\sim}{\mu}$ and $\underset{\sim}{\nu}$, and that the absorption probability $\phi_0(\underset{\sim}{y})$ is constant (equal to Π_0).

The analysis will be confined to slab geometry, so that the state $\underset{\sim}{y}$ will be represented by a pair of scalars (x, μ) and λ will be a function of x only. By a simple scale transformation, we may assume without loss of generality that λ is constant.

In this report, we shall use the discrete approximation method developed by Brockwell[3] to derive the equations giving the first two moments of the joint first-passage distribution for the model described above. That is, equations will be given for the means, variances, and covariances of the numbers of particles emerging with various states from the slab, conditional on a single initial particle in the slab with given initial state. In addition to calculating solutions of these equations, we shall also use them to estimate the critical length of the slab, i.e., the smallest length for which the mean number of emergent particles becomes infinite. (This is one of the many possible definitions of critical length. For a discussion of the relationship between the various definitions, see Moyal[11] and Brockwell and Moyal.[5])

The discrete approximation we shall use in the analysis is presented in Section II.

The development of the equations for the first two moments and the solutions of these equations are given in Section III.

A computer program was written to calculate the solutions, and the results are given for a variety of different collision outcome distributions in Section IV. The appendix gives in detail the mathematical methods used in the computer code to solve the equations.

II. THE BASIS OF THE DISCRETE APPROXIMATION

The icosahedral approximation developed by Brockwell[3] is used. A complete discussion of this approximation is given in Chapter VI of Ref. 3, so only a summary is given here. We are concerned with scattering, absorption, and fission in a plane slab, infinite in two directions and with finite thickness t.

The state of any particle is determined by its position and direction of motion. (We consider only the case of particles with a single energy, i.e., the one-group case, although there is no conceptual difficulty in the generalization involving multiple energies. In fact, H. Greenspan (of the ANL Applied Mathematics Division) is currently working on a computer program using analogs of the equations developed in Section III for particles with a finite set of possible energies. His results will provide the first- and second-moment structure for the more realistic case in which the dependence of λ and $\{\phi_n\}$ on energy, or other additional state variables, is taken into account.) The position of any particle is specified by its distance $x\,(0 \le x \le t)$ from the left face of the slab. It is in the specification of the direction of motion that the icosahedral approximation enters.

The essential approximation to the model described in Section I is the following: The directions of motion of each particle are constrained to a set of 30 unit vectors corresponding to points evenly spaced on the surface of the unit sphere. That is, a particle may not travel in an arbitrary direction, but only in directions represented by these 30 unit vectors.

The 30 vectors are chosen in the following way: A regular icosahedron is oriented as in Fig. 1 so that the line joining its center to the midpoint of one edge is in the direction of the unit vector $\underset{\sim}{i}$ normal to the slab faces (and in the direction of increasing x). The 30 possible directions of motion are then taken to be along the lines joining the center of the icosahedron to the midpoints of its edges. If we let $\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_{30}\}$ be the unit vectors in these directions, then the set of scalar products $\{\underset{\sim}{a}_i \cdot \underset{\sim}{a}_1, \underset{\sim}{a}_i \cdot \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_i \cdot \underset{\sim}{a}_{30}\}$ gives the cosines of the possible angular deflections for a particle initially traveling in direction $\underset{\sim}{a}_i$. This (unordered) set is the same for each i (i = 1, ..., 30). This means that for any direction of motion before a collision, the set of all possible angular deflections resulting from the collision is the same, independent of the initial direction of motion; that is, we can define a single set of 30 possible angular deflections $\{\epsilon_1, \ldots, \epsilon_{30}\}$ which take an arbitrary initial direction $\underset{\sim}{a}_i$ into the set $\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_{30}\}$ of all possible directions of motion. This set $\{\epsilon_1, \ldots, \epsilon_{30}\}$ contains only nine <u>distinct</u> angles $\theta_1, \ldots, \theta_9$, where

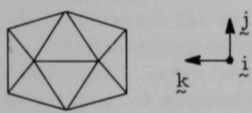

Fig. 1. Orientation of the Icosahedron

$$\begin{aligned}
\cos\theta_1 &= \mu_1, & \cos\theta_9 &= -\mu_1,\\
\cos\theta_2 &= \mu_2, & \cos\theta_8 &= -\mu_2,\\
\cos\theta_3 &= \mu_3, & \cos\theta_7 &= -\mu_3,\\
\cos\theta_4 &= \mu_4, & \cos\theta_6 &= -\mu_4,\\
\cos\theta_5 &= \mu_5, & &
\end{aligned} \tag{5}$$

and

$$
\begin{aligned}
\mu_1 &= 1 &&= -\mu_{10},\\
\mu_2 &= \tfrac{1}{4}\left(1+\sqrt{5}\right) &&= -\mu_9,\\
\mu_3 &= \tfrac{1}{2} &&= -\mu_8,\\
\mu_4 &= \tfrac{1}{4}\left(\sqrt{5}-1\right) &&= -\mu_7,\\
\mu_5 &= 0 &&= -\mu_6.
\end{aligned}
\tag{6}
$$

The angles $\theta_2, \theta_3, \ldots, \theta_8$ each occur four times in the set $\{\epsilon_1, \ldots, \epsilon_{30}\}$. Assuming that the probability of a deflection through angle θ depends only on θ, we need therefore define only nine distinct transition probabilities $\{p_1, 4p_2, \ldots, 4p_8, p_9\}$ corresponding to angular deflections $\{\theta_1, \theta_2, \ldots, \theta_9\}$. The problem of determining the p_i's from the distribution $\phi(d\underset{\sim}{\nu}|\underset{\sim}{\mu})$ defined in Eq. 3 is considered later in this section.

With the orientation of the icosahedron defined above, the cosines of the angles between the 30 vectors representing possible directions of motion and the positive x-axis take only nine distinct values, $\mu_1, \ldots, \mu_5$ $(=\mu_6)$, $\mu_7, \ldots, \mu_{10}$, where the μ_i's are defined in Eq. 6. Considerable simplification can therefore be achieved by grouping the directions of motion according to the cosine of the angle they make with the positive x-axis. The number $\alpha (\alpha = 1, \ldots, 10)$ will be used to denote the set of all directions of motion whose direction cosine is μ_α. The definition must be modified slightly for the directions 5 and 6 perpendicular to the x-axis. Direction 5 consists (see Fig. 1) of the two unit vectors $\pm\underset{\sim}{k}$, and direction 6 consists of the unit vectors $\pm\underset{\sim}{j}$. This splitting of the directions perpendicular to the x-axis is necessary so that the process, when defined in terms of the state variables (x, α), be Markovian. We say that a particle has state (x, α) if its distance from the left end of the slab is $x\ (0 \le x \le t)$, and the cosine of its direction of motion is μ_α $(\alpha = 1, \ldots, 10)$.

In practice, of course, the particle can move in any direction in space. When a particle makes a collision with an atom of the slab, it gives rise to a random number $(n = 0, 1, 2, \ldots)$ of particles of the same type. As described in Eq. 3, we shall assume that, conditional on a particle with direction $\underset{\sim}{\mu}$ experiencing a collision and giving rise to n particles $(n \ge 1)$, the directions of motion of the resulting particles are independently and identically distributed with distribution $\phi(d\underset{\sim}{\nu}|\underset{\sim}{\mu})$, where

$$\phi(d\underset{\sim}{\nu}|\underset{\sim}{\mu}) = \sigma(\theta)\, d\underset{\sim}{\nu}, \tag{7}$$

θ is the angle between $\underset{\sim}{\nu}$ and $\underset{\sim}{\mu}$, and $d\underset{\sim}{\nu}$ is an element of solid angle surrounding the unit vector $\underset{\sim}{\nu}$. We must determine $p_1, \ldots, p_9$ corresponding to the continuous function $\sigma(\theta)$. Since the original vectors $\{\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_{30}\}$ each correspond to congruent regions of the unit sphere, the probabilities are chosen so that

$$p_i = k\sigma(\theta_i), \quad i = 1, \ldots, 9, \tag{8}$$

the constant k being determined by the normalization condition,

$$p_1 + 4\sum_{i=2}^{8} p_i + p_9 = 1. \tag{9}$$

For example, if scattering is isotropic, then $\sigma(\theta) = 1/(4\pi)$, and Eqs. 8 and 9 give

$$p_1 = p_2 = \cdots = p_9 = \frac{1}{30}.$$

If a particle with direction of motion i ($i = 1, \ldots, 10$) makes a collision that results in the production of n particles ($n \geq 1$), then the directions of motion of the resulting particles are independently distributed, the distribution depending on the direction of motion of the colliding particle. This common distribution is specified (in the discrete model) by the probabilities

P_{ij} = Prob {the resulting particle emerges from the collision with direction j, given that the colliding particle was traveling in direction i just before the collision}.

These probabilities must satisfy the normalization condition,

$$\sum_{j=1}^{10} P_{ij} = 1, \quad i = 1, \ldots, 10. \tag{10}$$

The probabilities P_{ij} depend on which of the angular deflections $\theta_1, \ldots, \theta_9$ can bring about a transition from direction i to direction j, and this in turn depends on the geometry of the icosahedron. For example, a transition from direction 2 to direction 2 can occur by deflections through θ_1, θ_2, θ_3, or θ_4, and consequently $P_{22} = p_1 + p_2 + p_3 + p_4$. The other P_{ij}'s are found similarly and are shown below as the matrix P in which P_{ij} is the jth element of the ith row and the symbol $(i_1, i_2, \ldots, i_n)$ is used as an abbreviation for $p_{i_1} + \cdots + p_{i_n}$.

$$P = \begin{bmatrix} (1) & (2222) & (3333) & (4444) & (55) & (55) & (6666) & (7777) & (8888) & (9) \\ (2) & (1234) & (2345) & (2356) & (46) & (37) & (4578) & (5678) & (6789) & (8) \\ (3) & (2345) & (1267) & (2457) & (28) & (46) & (3568) & (3489) & (5678) & (7) \\ (4) & (2356) & (2457) & (1368) & (37) & (28) & (2479) & (3568) & (4578) & (6) \\ (5) & (4466) & (2288) & (3377) & (19) & (55) & (3377) & (2288) & (4466) & (5) \\ (5) & (3377) & (4466) & (2288) & (55) & (19) & (2288) & (4466) & (3377) & (5) \\ (6) & (4578) & (3568) & (2479) & (37) & (28) & (1368) & (2457) & (2356) & (4) \\ (7) & (5678) & (3489) & (3568) & (28) & (46) & (2457) & (1267) & (2345) & (3) \\ (8) & (6789) & (5678) & (4578) & (46) & (37) & (2356) & (2345) & (1234) & (2) \\ (9) & (8888) & (7777) & (6666) & (55) & (55) & (4444) & (3333) & (2222) & (1) \end{bmatrix} \qquad (11)$$

Note that each row contains p_1 and p_9 once and each of $p_2, \ldots, p_8$ four times. This is because each probability p_i corresponds to a particular unit vector representing the direction of motion of the particle after the collision, and, of the 30 possible unit vectors, exactly four correspond to each angular deflection $\theta_2, \ldots, \theta_8$. Thus Condition 9 implies Condition 10.

III. EQUATIONS FOR FIRST AND SECOND MOMENTS

In this section we develop the equations for the mean number of particles emerging in direction α ($\alpha = 1, \ldots, 4, 7, \ldots, 10$) conditional on an initial particle at position x ($0 \le x \le t$) within the slab and traveling in direction β ($\beta = 1, \ldots, 10$). Particles cannot emerge traveling in direction 5 or 6, as these directions are parallel to the faces of the slab. The equations are also derived giving the variance-covariance structure of the number of particles emergent in direction α_1 and α_2 ($\alpha_1 = 1, \ldots, 4, 7, \ldots, 10$; $\alpha_2 = 1, \ldots, 4, 7, \ldots, 10$) conditional on a particle at position x ($0 \le x \le t$) and traveling in direction β ($\beta = 1, \ldots, 10$). The solutions of these equations are then presented.

Denote by $\underset{\sim}{N} = (N_1, \ldots, N_4, N_7, \ldots, N_{10})$ the numbers of particles making first passages from the slab $0 \le x \le t$ in directions 1, ..., 4, 7, ..., 10, respectively, and let $P(\underset{\sim}{N}|x, \nu)$ be the probability that $\underset{\sim}{N}$ particles make first passages from the slab conditional on a single initial particle with position x and direction cosine ν. Similarly, let $P(\underset{\sim}{N}|x, \nu_1, \ldots, \nu_k)$ denote the probability that $\underset{\sim}{N}$ particles make first passages conditional on k initial particles with position x and direction cosines $\nu_1, \ldots, \nu_k$ ($-1 \le \nu_i \le 1$, $i = 1, \ldots, k$). It is convenient to introduce the generating functions

$$G(x, \nu) = \sum_{n_1, \ldots, n_4, n_7, \ldots, n_{10} = 0}^{\infty} P(\underset{\sim}{n}|x, \nu) z_1^{n_1} \ldots z_4^{n_4} z_7^{n_7} \ldots z_{10}^{n_{10}}, \quad |z_i| \le 1,$$

and

$$G(x, \nu_1, \ldots, \nu_k) = \sum_{n_1, \ldots, n_4, n_7, \ldots, n_{10}=0}^{\infty} P(\underline{n} \mid x, \nu_1, \ldots, \nu_k) z_1^{n_1} \ldots z_4^{n_4} z_7^{n_7} \ldots z_{10}^{n_{10}}, \ |z_i| \leq 1.$$

For clarity in writing G, we will suppress the arguments z_i. Then, since we are assuming that the particles do not interact with each other,

$$G(x, \nu_1, \ldots, \nu_k) = G(x, \nu_1) \ldots G(x, \nu_k).$$

The probability that a particle makes a collision in traveling a small distance δs is assumed to be $\lambda \delta s + o(\delta s)$, where λ is a positive constant. The probability of more than one collision is $o(\delta x)$. By measuring all distances in units of λ^{-1}, we can assume (without loss of generality) that $\lambda = 1$.

Given that a particle moving with direction cosine ν experiences a collision, we assume that there is a probability π_0 that the particle is absorbed and probabilities $\pi_k(d\nu_1, \ldots, d\nu_k \mid \nu)$, $k = 1, 2, \ldots$, that k particles are produced with direction cosines in $d\nu_1, \ldots, d\nu_k$ $(-1 \leq \nu_i \leq 1)$. In the discrete model, ν_i will take one of the values μ_α, $\alpha = 1, \ldots, 10$, with probability 1.

The initial particle will either experience no collisions before leaving the slab (in which case exactly one particle will emerge from the slab with direction cosine ν, or the particle will experience a first collision at some point y within the slab. By considering these two possibilities, we obtain the first collision equation for $G(x, \nu)$:

$$G(x, \nu) = e^{-\nu^{-1}(t-x)} \prod_{\alpha=1, \ldots, 4, 7, \ldots, 10} z_\alpha^{\delta_{\nu, \mu_\alpha}} + \int_{y=x}^{t} \pi_0 e^{-\nu^{-1}(y-x)} \nu^{-1} \, dy$$

$$+ \sum_{k=1}^{\infty} \int_{y=x}^{t} \int_{-1}^{1} \cdots \int_{-1}^{1} \pi_k(d\nu_1, \ldots, d\nu_k \mid \nu) G(y, \nu_1) \ldots G(y, \nu_k) e^{-\nu^{-1}(y-x)} \nu^{-1} \, dy, \quad \text{if } \nu > 0, \tag{12}$$

$$G(x, \nu) = e^{\nu^{-1} x} \prod_{\alpha=1, \ldots, 4, 7, \ldots, 10} z_\alpha^{\delta_{\nu, \mu_\alpha}} + \int_{y=0}^{x} \pi_0 e^{-\nu^{-1}(y-x)} \nu^{-1} \, dy$$

$$+ \sum_{k=1}^{\infty} \int_{y=0}^{x} \int_{-1}^{1} \cdots \int_{-1}^{1} \pi_k(d\nu_1, \ldots, d\nu_k \mid \nu) G(y, \nu_1) \ldots G(y, \nu_k) e^{-\nu^{-1}(y-x)} \nu^{-1} \, dy, \quad \text{if } \nu < 0. \tag{13}$$

($\delta_{\nu, \mu}$ is defined to be one if $\nu = \mu$, zero otherwise.) Differentiating Eqs. 12 and 13 with respect to x, we obtain the "backward" Kolmogorov equation,

$$\nu\frac{\partial}{\partial x}G(x,\nu) - G(x,\nu) = -\pi_0 - \sum_{k=1}^{\infty}\int_{-1}^{i}\cdots\int_{-1}^{1}\pi_k(d\nu_1\ldots d\nu_k|\nu)G(x,\nu_1)\ldots G(x,\nu_k), \tag{14}$$

with boundary conditions

$$G(x,\nu) = \prod_{\alpha=1,\ldots,4,7,\ldots,10} z_\alpha^{\delta_{\nu,\mu_\alpha}} \quad \begin{array}{l}\text{if } x = 0 \text{ and } \nu < 0,\\ \text{or } x = t \text{ and } \nu > 0.\end{array} \tag{15}$$

(These equations are in fact a special case of Eqs. 1 and 2.)

We now make the further assumption (in accordance with Eq. 3) that the probability measures π_k, $k \geq 1$, have the form

$$\pi_k(d\nu_1,\ldots,d\nu_k|\nu) = \pi_k\pi(d\nu_1|\nu)\pi(d\nu_2|\nu)\ldots\pi(d\nu_k|\nu), \tag{16}$$

where π_k is the probability that k particles are produced and

$$\int_{-1}^{1}\pi(d\nu_i|\nu) = 1.$$

The backward equation then becomes

$$\left[\mu\frac{\partial}{\partial x} - 1\right]G(x,\mu) = -\pi_0 - \sum_{k=1}^{\infty}\pi_k\left[\int_{-1}^{1}\pi(d\nu|\mu)G(x,\nu)\right]^k, \tag{17}$$

with the same boundary conditions as above.

Let $M_\alpha(x,\mu)$ denote the expected number of particles emerging from the slab with direction α (i.e., with direction cosine μ_α), $\alpha = 1, \ldots, 4, 7, \ldots, 10$. Then, under the assumption that M_α is bounded and

$$\sum_{0}^{\infty} k\pi_k = m < \infty,$$

we have

$$M_\alpha(x,\mu) = \left[\frac{\partial}{\partial z_\alpha}G(x,\mu)\right]_{z_\beta=1,\beta=1,\ldots,4,7,\ldots,10}. \tag{18}$$

Hence, from Eqs. 17 and 15,

$$\left[\mu\frac{\partial}{\partial x} - 1\right] M_\alpha(x,\mu) = -m \int_{-1}^{1} \pi(d\nu|\mu) M_\alpha(x,\nu), \tag{19}$$

and

$$M_\alpha(x,\mu) = \delta_{\mu,\mu_\alpha} \quad \begin{array}{l}\text{if } x = 0 \text{ and } \mu < 0, \\ \text{or } x = t \text{ and } \mu > 0.\end{array} \tag{20}$$

(In deriving Eq. 19 from Eqs. 17 and 18, we have used the fact that

$$[G(x,\mu)]_{z_\beta=1,\beta=1,\ldots,4,7,\ldots,10} = 1.$$

This follows from our assumption that M_α is bounded.)

The second factorial moments of the numbers of emergent particles are obtained by a further differentiation of the generating function G. Thus, if we define

$$S_{\alpha\beta}(x,\mu) = \begin{cases} EN_\alpha N_\beta & \alpha \neq \beta \\ E[N_\alpha(N_\alpha - 1)] & \beta = \alpha \end{cases},$$

(assuming again that these moments are bounded), then

$$S_{\alpha\beta}(x,\mu) = \left[\frac{\partial^2}{\partial z_\alpha \partial z_\beta} G(x,\mu)\right]_{z_1=\ldots=z_4=z_7=\ldots=z_{10}=1} . \tag{21}$$

Provided

$$\sum_{k=0}^{\infty} k(k-1)\pi_k = m_{(2)} < \infty,$$

we then have, from Eqs. 15, 17, and 21,

$$\left[\mu\frac{\partial}{\partial x} - 1\right] S_{\alpha\beta}(x,\mu) = -m \int_{-1}^{1} \pi(d\nu|\mu) S_{\alpha\beta}(x,\nu)$$

$$-m_{(2)}\left\{\int_{-1}^{1} \pi(d\nu|\mu) M_\alpha(x,\nu)\right\}\left\{\int_{-1}^{1} \pi(d\nu|\mu) M_\beta(x,\nu)\right\}. \tag{22}$$

and

$$S_{\alpha\beta}(x,\mu) = 0 \quad \text{if} \quad x = 0 \text{ and } \mu < 0, \quad \text{or } x = t \text{ and } \mu > 0. \tag{23}$$

(Equations 22 and 23 are valid whether or not $\alpha = \beta$.)

So far we have not fully exploited our discrete approximation in deriving the equations for M_α and $S_{\alpha\beta}$. In the discrete approximation, the initial direction μ can take only one of the values $\{\mu_1, \ldots, \mu_{10}\}$ defined in Section II, and the integrals with respect to the measure π are in fact sums. Thus,

$$\int_{-1}^{1} M_\alpha(x,\nu)\pi(d\nu|\mu_i) = \sum_{j=1}^{10} M_\alpha(x,\mu_j)P_{ij},$$

where P_{ij} is the probability that a particle produced at a collision travels in direction j conditional on the colliding particle traveling in direction i before the collision. Thus P_{ij} is the (i, j)th element of the matrix given in Eq. 11. Similarly,

$$\int_{-1}^{1} S_{\alpha\beta}(x,\nu)\pi(d\nu|\mu_i) = \sum_{j=1}^{10} S_{\alpha\beta}(x,\mu_j)P_{ij}.$$

It is now convenient to change notation slightly. We write $M_\alpha(x,i)$ for $M_\alpha(x,\mu_i)$ $(i = 1, \ldots, 10)$. Thus $M_\alpha(x,i)$ is the mean number of particles emergent in direction α $(\alpha = 1, \ldots, 4, 7, \ldots, 10)$, conditional on one particle at position x $(0 \leq x \leq t)$ traveling in direction i. The cosine of the angle between the direction of motion i and the positive x-axis is therefore μ_i. Similarly, we write $S_{\alpha\beta}(x,i)$ for $S_{\alpha\beta}(x,\mu_i)$ $(i = 1, \ldots, 10)$. Thus Eq. 19 may be written in matrix form,

$$\begin{bmatrix} \mu_1 \frac{\partial}{\partial x} - 1 & & 0 \\ & \ddots & \\ 0 & & \mu_{10}\frac{\partial}{\partial x} - 1 \end{bmatrix} \begin{bmatrix} M_\alpha(x,1) \\ \vdots \\ M_\alpha(x,10) \end{bmatrix} = -m \begin{bmatrix} P_{1\,1} & \ldots & P_{1\,10} \\ \vdots & & \\ P_{10\,1} & & P_{10\,10} \end{bmatrix} \cdot \begin{bmatrix} M_\alpha(x,1) \\ \vdots \\ M_\alpha(x,10) \end{bmatrix}. \tag{24}$$

Since $\mu_5 = \mu_6 = 0$, we can write $M_\alpha(x,5)$ and $M_\alpha(x,6)$ as linear functions of $M_\alpha(x,1)$, ..., $M_\alpha(x,4)$, $M_\alpha(x,7)$, ..., $M_\alpha(x,10)$. Doing this, and rearranging the terms, we may express Eq. 24 as

$$\frac{\partial}{\partial x}\begin{bmatrix} M_\alpha(x,1) \\ \cdot \\ \cdot \\ M_\alpha(x,4) \\ M_\alpha(x,7) \\ \cdot \\ \cdot \\ M_\alpha(x,10) \end{bmatrix} = R \begin{bmatrix} M_\alpha(x,1) \\ \cdot \\ \cdot \\ M_\alpha(x,4) \\ M_\alpha(x,7) \\ \cdot \\ \cdot \\ M_\alpha(x,10) \end{bmatrix}, \tag{25}$$

where R is an 8 x 8 matrix which can be determined from Eq. 24. This is a system of eight homogeneous first-order linear differential equations, with boundary conditions given by Eq. 20. The solution is

$$\begin{bmatrix} M_\alpha(x,1) \\ \cdot \\ \cdot \\ \cdot \\ M_\alpha(x,4) \\ M_\alpha(x,7) \\ \cdot \\ \cdot \\ \cdot \\ M_\alpha(x,10) \end{bmatrix} = e^{RX} \cdot \underset{\sim}{c}, \tag{26}$$

where $\exp(Rx)$ is the 8 x 8 matrix defined as

$$\sum_{k=0}^{\infty} (Rx)^k/k!,$$

and $\underset{\sim}{c}$ is an eight-component column vector determined by the boundary conditions.

In fact, if we let $M(x)$ be the 8 x 8 matrix

$$\begin{bmatrix} M_1(x,1) & \dots & M_4(x,1) & M_7(x,1) & \dots & M_{10}(x,1) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ M_1(x,4) & \dots & M_4(x,4) & M_7(x,4) & \dots & M_{10}(x,4) \\ M_1(x,7) & \dots & M_4(x,7) & M_7(x,7) & \dots & M_{10}(x,7) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ M_1(x,10) & \dots & M_4(x,10) & M_7(x,10) & \dots & M_{10}(x,10) \end{bmatrix}, \tag{27}$$

then

$$M(x) = e^{Rx} \cdot C, \tag{28}$$

where C is an 8 x 8 matrix determined by the boundary conditions. Using the boundary conditions for $x = 0$ to get the lower half of C, and for $x = t$ to get the upper half of $M(t)$, we have (by matrix partitioning)

$$\left[\begin{array}{c|c} I & 0 \\ \hline A & B \end{array}\right] = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} C_1 & C_2 \\ \hline 0 & I \end{array}\right], \tag{29}$$

where

$$\left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array}\right] = \exp(Rt),$$

and each of the indicated smaller matrices above is 4 x 4. Thus, $C_1 = E_{11}^{-1}$ and $C_2 = -E_{11}^{-1} \cdot E_{12}$. Since $M_\alpha(x, 5)$ and $M_\alpha(x, 6)$ ($\alpha = 1, \ldots, 4, 7, \ldots, 10$) can be expressed as linear functions of the terms in $M(x)$, the above gives a complete solution for the mean number of particles emergent in any of the eight possible emergent directions, conditional on one particle at an arbitrary position x within the slab, traveling in any of the 10 possible directions.

The above equations implicitly assume that the mean number of particles produced is finite. If the slab is supercritical, i.e., if the length of the slab is such that the mean of the total number of particles produced is infinite, then these equations will not be valid. However, this method may be used to determine the critical length for a given m

$$\left(= \sum_{k=0}^{\infty} k\pi_k \right)$$

by varying the length t of the slab, and seeing for which value of t the total mean number of particles produced approaches infinity. This has been done, and the results are reported in Section IV.

To find the functions $S_{\alpha\beta}$, we now apply similar methods to those used above. Employing the relationship between $\pi(d\nu | \mu)$ and the matrix P, we may write Eq. 22 as

$$\begin{bmatrix} \mu_1 \frac{\partial}{\partial x} - 1 & & 0 \\ & \ddots & \\ 0 & & \mu_{10} \frac{\partial}{\partial x} - 1 \end{bmatrix} \begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} = -m \begin{bmatrix} P_{11} & \dots & P_{1\,10} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ P_{10\,1} & \dots & P_{10\,10} \end{bmatrix} \cdot \begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix}$$

$$-m_{(2)} \begin{bmatrix} F_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ F_{\alpha\beta}(x,10) \end{bmatrix} \qquad (\alpha = 1, \dots, 4, 7, \dots, 10;\ \beta = 1, \dots, 4, 7, \dots, 10), \tag{30}$$

where $F_{\alpha\beta}(x,i) = \left\{ \sum_{j=1}^{10} M_\alpha(x,j) P_{ij} \right\} \cdot \left\{ \sum_{j=1}^{10} M_\beta(x,j) \cdot P_{ij} \right\} \quad (i = 1, \dots, 10).$

But, since $\mu_5 = \mu_6 = 0$, we can again write $S_{\alpha\beta}(x,5)$ and $S_{\alpha\beta}(x,6)$ as linear functions of $S_{\alpha\beta}(x,i)$ $(i = 1, \dots, 4, 7, \dots, 10)$ and $F_{\alpha\beta}(x,i)$ $(i = 5, 6)$. Then,

$$\frac{\partial}{\partial x} \begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,4) \\ S_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} = R \begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,4) \\ S_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} + \begin{bmatrix} V_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ V_{\alpha\beta}(x,4) \\ V_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ V_{\alpha\beta}(x,10) \end{bmatrix}, \tag{31}$$

where R is the same 8 x 8 matrix that appeared in Eq. 25, and $V_{\alpha\beta}(x,i)$ incorporates $m_{(2)}$ and is a linear function of $F_{\alpha\beta}(x,i)$, $F_{\alpha\beta}(x,5)$, and $F_{\alpha\beta}(x,6)$ $(i = 1, \dots, 4, 7, \dots, 10)$.

Here we have a set of eight first-order, linear, nonhomogeneous differential equations. The solution is analogous to the solution of a first-order, linear, nonhomogeneous differential equation,

$$\begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,4) \\ S_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} = e^{Rx} \cdot \underset{\sim}{c} + \int_0^x e^{R(x-y)} \cdot \underset{\sim}{V}_{\alpha\beta}(y)\, dy, \tag{32}$$

where $\underset{\sim}{V}_{\alpha\beta}(y)$ is the eight-component vector given in Eq. 31. The eight-component vector $\underset{\sim}{c}$ is determined by the boundary conditions in Eq. 23. In fact, since $\underset{\sim}{S}_{\alpha\beta}(0) = \underset{\sim}{c}$, the bottom four elements of $\underset{\sim}{c}$ are zeros. The top four elements are determined using the fact that the top four elements of $\underset{\sim}{S}_{\alpha\beta}(t)$ are zeros.

The covariances, variances, and correlations can then be calculated from the $S_{\alpha\beta}$'s and the M_α's by the appropriate formulas. The appendix gives the mathematical methods and details used in writing the computer program that determines the M_α's and the $S_{\alpha\beta}$'s.

IV. NUMERICAL RESULTS

The numerical results concerning the first- and second-moment structure of the first-passage distribution are presented in this section. First, however, is a discussion of the accuracy of the approximate model used, and of the methods used to verify that the program was coded correctly.

This model does not permit particles to take arbitrary directions of motion within the slab; only the 30 directions corresponding to the midpoints of the edges of an icosahedron are permitted. The question of the accuracy of results from this model, relative both to exact results and to results from models using different approximations, is discussed by Brockwell,[3] Chapter VII, Sections 5 and 6. For problems in slab geometry involving only scattering and absorption, exact solutions are possible in terms of the X- and Y-functions of Chandrasekhar.[7] Numerical comparisons given in Ref. 3 show that the icosahedral approximation compares favorably with the Gaussian quadrature approximation with $n = 4$ and the spherical harmonics method with $n = 3$. This suggests that the icosahedral approximation should give accurate results for other cases in which the exact solutions are not known.

Internal symmetries in the model were used to verify that the coding was done correctly, that is, that the numerical output corresponds to the mathematical analysis. Considering the symmetry inherent in our model, we see that $M_\alpha(x, \ell) = M_{11-\alpha}(t - x, 11 - \ell)$ $(0 \le x \le t;\ \alpha = 1, \ldots, 4, 7, \ldots, 10;$ $\ell = 1, \ldots, 10)$. The output for the M_α's conditional on a particle at position x was compared with the output conditional on a particle at position t - x $(0 \le x \le t)$, and it was verified that the required symmetries did in fact hold for the numerical output. Similarly, $S_{\alpha\beta}(x, \ell) = S_{\beta\alpha}(x, \ell)$, and $S_{\alpha\beta}(x, \ell) = S_{11-\alpha, 11-\beta}(t - x, 11 - \ell)$ $(\alpha = 1, \ldots, 4, 7, \ldots, 10;\ \beta = 1, \ldots, 4, 7, \ldots, 10;\ 0 \le x \le t;$ $\ell = 1, \ldots, 10)$. The $S_{\alpha\beta}(x, \ell)$'s were calculated for $\alpha = 1, \ldots, 4;$ $\beta = 1, \ldots, 4, 7, \ldots, 10;$ and $\ell = 1, \ldots, 4, 7, \ldots, 10$. The required symmetries were again satisfied by the numerical output.

A. Calculations of the Critical Length

We shall take the critical length t_c of a slab to be the smallest length for which the mean Eqs. 19 and 20 have no bounded nonnegative solution; t_c depends on m, the mean number of particles produced per collision, and also upon the scattering law $\sigma(\theta)$. (We have assumed this applies to fission products as well as to truly scattered particles; it would not be difficult to extend our method to the case where one angular distribution applies to scattered particles and another to fission-produced particles.) Now, as we increase the slab thickness (starting from t = 0), the mean number of emergent particles m_{tot} (conditional on a single initial particle incident normally upon the slab) increases, approaching ∞ as $t \uparrow t_c$. If t is increased slightly beyond t_c, the mean numbers of emergent particles in some directions (as calculated from Eqs. 19 and 20 become negative. Since the difference between the smallest t giving negative means and the largest t for which we can find large positive solutions of Eqs. 19 and 20 is small, we can estimate t_c fairly accurately (within the limits of the icosahedral model) from the numerical results. This was done for two scattering laws—isotropic scattering and Rayleigh scattering (the latter being used primarily to give an idea of the sensitivity of the results to deviations from isotropy).

For isotropic scattering, $\sigma(\theta) = 1/(4\pi)$; therefore the directions of motion immediately after a collision are uniformly distributed over the unit sphere, and independent of the direction of motion immediately before the collision. This scattering law is often used as a reasonable approximation for neutron multiplication. For Rayleigh scattering, $\sigma(\theta) = 3(1 + \cos^2 \theta)/(16\pi)$. With this scattering law, particles emerging from a collision are more likely to move in a forward or backward direction than perpendicular to the direction of motion of the colliding particle. The Rayleigh scattering law is often used when studying the passage of photons through stellar atmospheres. Other scattering laws relevant to different physical situations can be analyzed without essential modification of the computer program.

Table I gives the critical length t_c for several values of m for both isotropic and Rayleigh scattering. The distance t_c is measured in units of the mean free path $1/\lambda$. No method is known that will give the exact value of t_c for either isotropic or Rayleigh scattering. (The method of Case[6] can in principle be used to give any required degree of accuracy, but the analysis becomes extremely complicated as the degree of accuracy is increased.) However, Mullikin[12] has given simple formulas that may be used to calculate upper and lower bounds for t_c for isotropic scattering and any value of m. A computer program was written to calculate these bounds for different m, and the results are presented in columns 4 and 5 of Table I. The final two columns of Table I present critical lengths that Case[6] and Mitsis[8] have obtained for isotropic scattering using different approximate methods. The blanks in these two columns indicate that numerical results were not reported for some values of m.

TABLE I. Dependence of the Critical Length on m (except for column 2, all results are for isotropic scattering)

	t_c		Mullikin's Bounds[12]			
m	Rayleigh	Isotropic	Lower	Upper	Case[6]	Mitsis[8]
1.05	6.589	6.605	6.553	6.684		
1.10	4.214	4.233	4.191	4.286	4.227	4.24
1.30	1.866	1.887	1.857	1.905	1.875	1.780
1.50	1.201	1.221	1.199	1.227	1.209	
1.60	1.014	1.034	1.014	1.039	1.023	1.022
1.80	0.764	0.784	0.771	0.792	0.785	
2.00	0.605	0.623	0.617	0.639	0.640	0.621
2.20	0.493	0.511	0.511	0.534		
2.40	0.411	0.429	0.434	0.459		
2.60	0.348	0.365	0.376	0.403		

Our computations of t_c for isotropic scattering fall within Mullikin's bounds for $m \leq 2.2$. However, for $m > 2.2$, our t_c is slightly below the lower bound of Mullikin. This indicates that our model should be reasonably accurate for moderate values of m, but for larger m, corresponding to smaller critical lengths, the accuracy of the approximation decreases. Comparing the critical lengths computed by Case and Mitsis, we see that Case's method gives a critical length outside Mullikin's bounds for m = 2.0. However, for smaller m, the lengths are within Mullikin's bounds. Mitsis' result for m = 1.3 is outside the bounds, but the other lengths he reported are within the bounds. Since neither Case nor Mitsis presented values for $m > 2.0$, comparison between the approximations in this range is difficult. Case's method can, in principle, be calculated to any desired accuracy; however, the zeroth and first-order approximation for which numerical results have been given depend for their accuracy upon m - 1 being small. This assumption, added to the fact that Case's critical length for m = 2.0 is outside Mullikin's bounds, while our method gives results within the bounds for m up to 2.2, suggests that our method is more accurate than Case's for m > 2.0. Mitsis' approximation is a modification of Case's and appears to be more accurate than Case's for large m. The values of t_c for Rayleigh scattering are close to, but always slightly less than, the values for isotropic scattering with the same m. The difference between the two lengths stays close to 0.018 as m varies.

B. The Mean Number of Emergent Particles

From the numerical solution of Eqs. 19 and 20, we obtain the mean number of particles $M_j(x, i)$ emergent in any direction μ_j, conditional on a single initial particle with position x and direction μ_i. In the continuous model, the mean number of particles emerging with direction cosines in $(\mu, \mu + d\mu)$ can be written as $f(\mu)\, d\mu$, where $f(\mu)$ is the mean density in direction μ. The function $f(\mu)$ can be estimated from the means $M_j(x, i)$ in the manner described in Ref. 3, p. 99.

Thus, for example, if we consider a single particle incident normally on the right face of the slab (i.e., if we take $x = t$, $\mu_i = -1$), then the mean density $f(\mu)$, $0 \le \mu \le 1$, of the reflected particles is estimated by

$$f(1) = 3.75 M_1(t, -1),$$

$$f(\mu_i) = 15 M_i(t, -1), \quad i = 2, 3, 4,$$

and

$$f(0) = 0.$$

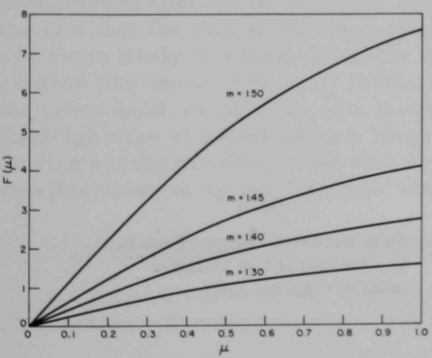

Fig. 2. The Mean Density of Particles Reflected from a Slab of Thickness 1.1 Mean Free Paths When a Single Particle Is Incident Normally on One Face

This function is illustrated in Fig. 2, where it is plotted for t = 1.1 and m = 1.3, 1.4, 1.45, and 1.50; scattering is assumed to be isotropic. As m approaches the value for which the slab becomes critical (m ≈ 1.55), the mean density begins to increase rapidly.

It is of interest to know how the total number, m_{tot}, of emergent particles varies with t for fixed m and, in particular, the rate at which m_{tot} approaches ∞ as $t \uparrow t_c$. This relationship is shown (again for a single initial particle normally incident on one slab face) for the values m = 1.1, 1.5, 2.0, and 2.473 in Figs. 3, 4, and 5. The results are for isotropic scattering; those for Rayleigh scattering are similar. Table II compares the results for Rayleigh and isotropic scattering m = 1.05. It

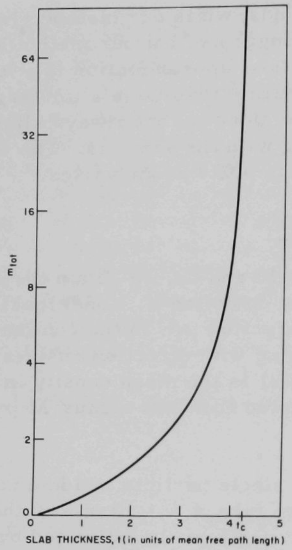

Fig. 3. Mean Number of Emergent Particles as a Function of Slab Thickness When m = 1.1

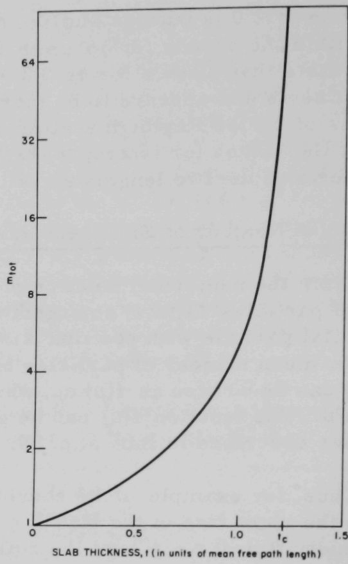

Fig. 4. Mean Number of Emergent Particles as a Function of Slab Thickness When m = 1.5

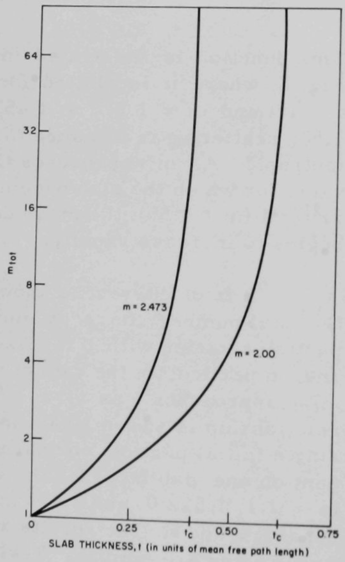

Fig. 5

Mean Number of Emergent Particles as a Function of Slab Thickness When m = 2.0 and 2.473

TABLE II. The Mean Number m_{tot} of Emergent Particles When the Mean Number m of Particles per Collision Is 1.05 and the Initial Particle Is Normally Incident on the Slab

Slab Thickness, t	Rayleigh Scattering	Isotropic Scattering	Slab Thickness, t	Rayleigh Scattering	Isotropic Scattering
0.1	1.006	1.006	4.3	1.940	1.943
0.3	1.020	1.020	4.5	2.068	2.069
0.5	1.036	1.037	4.7	2.221	2.221
0.7	1.054	1.056	4.9	2.409	2.407
0.9	1.075	1.077	5.1	2.645	2.639
1.1	1.097	1.101	5.3	2.952	2.941
1.3	1.122	1.126	5.5	3.369	3.350
1.5	1.149	1.153	5.7	3.970	3.935
1.7	1.178	1.182	5.9	4.916	4.849
1.9	1.210	1.214	6.1	6.630	6.482
2.1	1.244	1.248	6.3	10.702	10.249
2.3	1.280	1.285	6.5	32.992	28.389
2.5	1.320	1.325	6.55	74.04	
2.7	1.364	1.369	6.587	1,640	
2.9	1.411	1.416	6.589	15,013	
3.1	1.463	1.468	6.589+	∞	
3.3	1.520	1.525	6.6		626.5
3.5	1.585	1.589	6.603		1,936
3.7	1.656	1.661	6.604		9,002
3.9	1.738	1.742	6.604+		∞
4.1	1.831	1.835			

shows that for small values of t, m_{tot} is larger for isotropic scattering, but for larger values of t, m_{tot} is larger for Rayleigh scattering. This phenomenon appears to occur for all values of m and is probably due to the fact that for thin slabs the particles resulting from the first collision are more likely to escape from the slab in the Rayleigh case, owing to the greater likelihood of directly forward or directly backward scattering. On the other hand, as the slab gets thicker, there is more chance in the Rayleigh case of establishing a large self-sustaining population of particles moving nearly parallel to the slab faces. This would also explain the smaller value of t_c for Rayleigh scattering.

C. Calculations of the Second Moments

We now consider the second-moment structure of the first-passage distribution. Three parameters are under consideration:

$$m \left(= \sum_{n=0}^{\infty} n\pi_n \right),$$

the mean number of particles produced per collision;

$$\sigma^2 \left(= \sum_{n=0}^{\infty} (n = m)^2 \pi_n \right),$$

the variance of the number of particles produced per collision; and t, the length of the slab. The second-moment structure varies considerably with the values of these parameters. However, the behavior is essentially the same for Rayleigh as for isotropic scattering.

Equations 30-32 show that the second factorial moments are of the form

$$S_{\alpha\beta}(x, i) = m_{(2)} w_{\alpha\beta}(x, i),$$

where, for given $m, w_{\alpha\beta}(x, i)$ is independent of $\sigma^2 = m_{(2)} + m - m^2$. The covariance of the numbers of particles emergent in directions α and β is given by

$$\mathrm{Cov}_{\alpha\beta}(x, i) = S_{\alpha\beta}(x, i) - M_\alpha(x, i) M_\beta(x, i), \qquad \beta \neq \alpha;$$

$$\mathrm{Cov}_{\alpha\beta}(x, i) = S_{\alpha\alpha}(x, i) + M_\alpha(x, i) - M_\alpha(x, i)^2, \qquad \beta = \alpha.$$

Hence, for fixed m, t, x, and i, the covariance matrix of the numbers of particles emergent in the eight possible directions of motion is a linear function of σ^2. (If m is not an integer, σ^2 has a positive minimum possible value, which can easily be calculated.) In particular, the variance σ^2_{tot} of the <u>total</u> number of emergent particles is a linear function of σ^2; i.e.,

$$\sigma^2_{tot} = \xi + \eta\sigma^2,$$

where ξ and η depend only on m, t, x, and i.

To illustrate the numerical results, we shall again restrict ourselves to isotropic scattering with a single initial particle incident normally on one face of the slab. More precisely, we shall take $x = 0$ and $i = 1$ (recalling from Eq. 6 that $\mu_1 = 1$).

If $\pi_n = 0$ for $n \geq 2$, the second-moment structure of the emergent population is completely determined by the first-moment structure. In fact, the covariance of the numbers emerging in directions α and β is expressed in terms of the means as follows:

$$\mathrm{Cov}_{\alpha\beta} = -M_\alpha M_\beta \quad \text{if } \beta \neq \alpha;$$

$$\mathrm{Cov}_{\alpha\beta} = M\alpha - M_\alpha^2 \quad \text{if } \beta = \alpha.$$

We shall therefore not discuss this case any further and assume from now on that

$$\sum_{n=2}^{\infty} \pi_n > 0.$$

A typical example of the behavior when $m < 1$ is given by the case $\pi_0 = 0.95$, $\pi_{10} = 0.05$, for which $m = 0.50$, $\sigma = 2.1794$. The mean and the standard deviation of the number of transmitted particles both converge to 0 as $t \to \infty$; however, the mean and the standard deviation of the number of particles reflected approach nonzero limits as $t \to \infty$. Figure 6 shows the dependence of m_{tot} and σ_{tot}/m_{tot} on the slab thickness, t. Note that although m_{tot} is a decreasing function of t, σ_{tot}/m_{tot} increases to its asymptotic value. For $t = 0.25$, the correlations between emergent direction 1 and the other seven possible emergent directions are small in absolute value (≤ 0.10) and negative; all the other correlations are positive and in the range 0.35 to 0.60. This is due to the fact that if a particle emerges in direction 1 there is some indication that no collision has occurred and hence that no other particles emerge from the slab. However, if a particle emerges traveling in a direction other than 1, then a collision must have occurred and there are probably particles emerging in different directions. For $t = 0.50$, the same qualitative relation exists between the correlations, although all correlations are smaller in absolute value. For larger values of t, such as $t = 2.5$, all correlations are positive, although the correlations involving direction 1 are still smaller than the others; the largest element is 0.58. For the asymptotic case $t = \infty$ (which, as far as the reflected particles are concerned, seems to be essentially reached when $t = 4$), the correlations between directions 7, 8, 9, and 10 are all between 0.35 and 0.60.

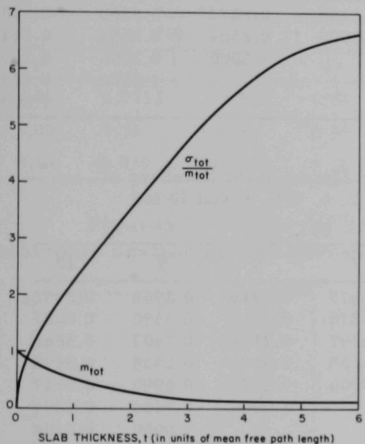

Fig. 6. The Mean and the Coefficient of Variation of the Number of Emergent Particles When $m = 0.5$, $\sigma = 2.1794$

We now turn to the case $m > 1$. The following describes the general behavior of the correlations for given m and σ as t increases from zero to t_c. For small t, the correlations are all positive, except for those involving direction 1, which are negative (for the reason explained in the preceding paragraph when $m < 1$). As t increases, the correlations all increase until, for some value between 0 and t_c, they are all positive. As t gets extremely close to t_c, all the correlations approach 1. To illustrate these effects, Tables III, IV, and V present the correlation matrices for isotropic scattering with $m = 1.5$ and $\sigma = 0.50$; the values of t are 0.5, 0.9, and 1.2 (the critical length t_c is 1.221).

TABLE III. Upper Half of the Correlation Matrix of the Numbers of Particles Emerging in Directions 1, 2, 3, 4, 7, 8, 9, and 10 for Slab Thickness $t = 0.5$

	1	2	3	4	7	8	9	10
1	1.0000	-0.3524	-0.3296	-0.2969	-0.3133	-0.3412	-0.3602	-0.2100
2		1.0000	0.2942	0.2697	0.2542	0.2835	0.3037	0.1778
3			1.0000	0.2575	0.2360	0.2646	0.2844	0.1667
4				1.0000	0.2100	0.2373	0.2563	0.1505
7					1.0000	0.2524	0.2651	0.1543
8						1.0000	0.2906	0.1694
9							1.0000	0.1797
10								1.0000

TABLE IV. Upper Half of the Correlation Matrix of the Numbers of Particles Emerging in Directions 1, 2, 3, 4, 7, 8, 9, and 10 for Slab Thickness $t = 0.9$

	1	2	3	4	7	8	9	10
1	1.0000	0.3082	0.3001	0.2849	0.2575	0.2815	0.2963	0.2352
2		1.0000	0.7532	0.7105	0.6870	0.7383	0.7690	0.6087
3			1.0000	0.6903	0.6597	0.7102	0.7407	0.5866
4				1.0000	0.6155	0.6642	0.6939	0.5498
7					1.0000	0.6771	0.6990	0.5517
8						1.0000	0.7458	0.5891
9							1.0000	0.6108
10								1.0000

TABLE V. Upper Half of the Correlation Matrix of the Numbers of Particles Emerging in Directions 1, 2, 3, 4, 7, 8, 9, and 10 for Slab Thickness $t = 1.2$

	1	2	3	4	7	8	9	10
1	1.0000	0.9977	0.9975	0.9972	0.9970	0.9975	0.9977	0.9965
2		1.0000	0.9989	0.9985	0.9983	0.9988	0.9990	0.9978
3			1.0000	0.9983	0.9981	0.9985	0.9988	0.9976
4				1.0000	0.9977	0.9981	0.9984	0.9972
7					1.0000	0.9983	0.9985	0.9973
8						1.0000	0.9989	0.9977
9							1.0000	0.9978
10								1.0000

The close resemblance between Eqs. 19 and 20 for $M_\alpha(x, i)$ and Eqs. 22 and 23 for $S_{\alpha\beta}(x, i)$ suggests that the second moments become infinite for the same length t_c as the first moments. It is of interest to examine how σ_{tot} changes as m, σ, and t vary. As discussed above, $\sigma^2_{tot} = \xi + \eta\sigma^2$, where (for fixed x and i, in particular, for the values x = 0, i = 1, under consideration) ξ and η depend only on m and t. The quantity ξ may be thought of as the variance due to the variability in the particle paths, and $\eta\sigma^2$ as the variance due to the variability in the number of particles produced per collision. Table VI gives the quantities ξ and η for several values of m and t. (We note again that for nonintegral m, there is a smallest possible positive value of σ, which can easily be calculated.)

TABLE VI. Values of ξ and η in the Relation $\sigma^2_{tot} = \xi + \eta\sigma^2$ for a Single Particle Incident Normally on a Slab of Length t

m = 1.10			m = 1.50			m = 2.00		
t	ξ	η	t	ξ	η	t	ξ	η
1.00	0.0712	2.752	0.50	1.612	3.434	0.25	3.348	2.166
3.00	7.32	102.8	0.90	39.74	65.01	0.50	212	115.5
4.15	40,010	372,180	1.20	203,700	275,230	0.60	39,456	20,050

Figures 7-9 show the relationship between σ_{tot}/m_{tot} and t for isotropic scattering and several values of m and σ. The case m = 2.4730, σ = 1.1051, corresponds to a probability distribution $\{\pi_n\}$ reported for neutron fission in U^{235}.[15] The graphs all show an inflection point for small values of t. The results indicate that $\sigma_{tot}/m_{tot} \to \infty$ as $t \uparrow t_c$; moreover, σ_{tot}/m_{tot} reaches the value 1 for t much smaller than t_c. Thus, even for a slab whose length is small compared with t_c, the variability in the number of emergent particles is quite large. This conclusion is related to the fact that, even with no multiplication of particles (i.e., $\pi_n = 0$ for $n \geq 2$), the variance of the number of collisions experienced by emergent particles is large (c.f. Abu-Shumays[1] and Brockwell[3]).

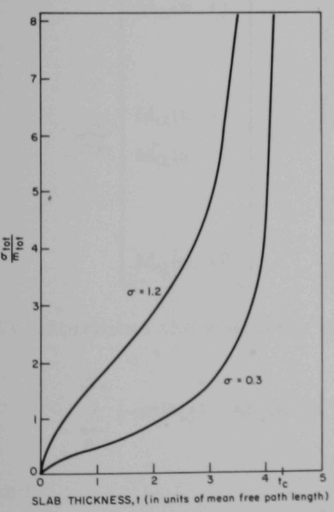

Fig. 7. The Coefficient of Variation of the Number of Emergent Particles When m = 1.1

The graphs corresponding to those in Figs. 7-9 for Rayleigh scattering are similar to the curves for isotropic scattering and are not shown. Of course, $\sigma_{tot}/m_{tot} \to \infty$ in this case as $t \uparrow t_c$ for Rayleigh scattering, not the value of t_c for isotropic scattering. The similarity of the two sets of curves indicates that the behavior of σ_{tot}/m_{tot} is insensitive to small changes in the scattering law.

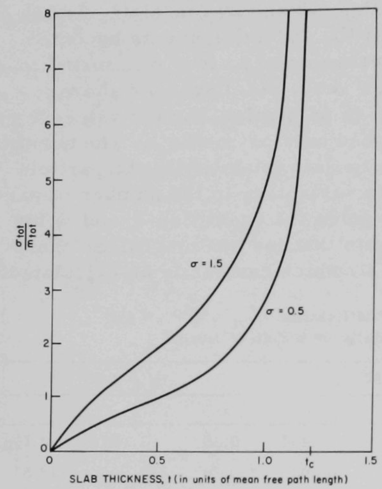

Fig. 8. The Coefficient of Variation of the Number of Emergent Particles When m = 1.5

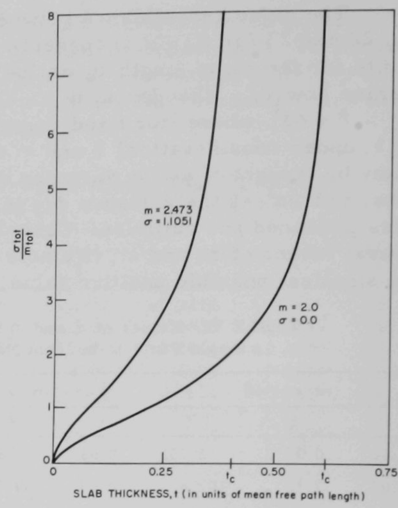

Fig. 9. The Coefficient of Variation of the Number of Emergent Particles When m = 2.0 and 2.473

APPENDIX A

Mathematical Methods Used in the Computer Program

This appendix presents in detail the mathematical methods used in the computer program to calculate the formulas for the means and the second factorial moments; these formulas were given in Eqs. 26 and 32.

First we discuss the computation of $M_\alpha(x, i)$, the mean number of particles emergent in direction α ($\alpha = 1, \ldots, 4, 7, \ldots, 10$) conditional on a particle at position x ($0 \le x \le t$) traveling in direction i ($i = 1, \ldots, 10$). We return to the basic Eq. 24; i.e.,

$$\begin{bmatrix} \mu_1 \frac{\partial}{\partial x} & & 0 \\ & \ddots & \\ 0 & & \mu_{10} \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} M_\alpha(x, 1) \\ \vdots \\ M_\alpha(x, 10) \end{bmatrix} = \begin{bmatrix} 1-mP_{1,1} & -mP_{1,2} & \cdots & -mP_{1,10} \\ \vdots & & & \vdots \\ -mP_{10,1} & & \cdots & 1-mP_{10,10} \end{bmatrix} \begin{bmatrix} M_\alpha(x, 1) \\ \vdots \\ M_\alpha(x, 10) \end{bmatrix}.$$

Since $\mu_5 = \mu_6 = 0$, we can rewrite this as Eq. 25,

$$\frac{\partial}{\partial x} \begin{bmatrix} M_\alpha(x, 1) \\ \vdots \\ M_\alpha(x, 4) \\ M_\alpha(x, 7) \\ \vdots \\ M_\alpha(x, 10) \end{bmatrix} = R \begin{bmatrix} M_\alpha(x, 1) \\ \vdots \\ M_\alpha(x, 4) \\ M_\alpha(x, 7) \\ \vdots \\ M_\alpha(x\ 10) \end{bmatrix}.$$

To determine the elements of the 8 x 8 matrix R, we use the equations

$$\sum_{j=1}^{10} (-mP_{5j}) \cdot M_\alpha(x, j) + M_\alpha(x, 5) = 0 \qquad \text{(A.1)}$$

and

$$\sum_{j=1}^{10} (-mP_{6j}) \cdot M_\alpha(x, j) + M_\alpha(x, 6) = 0 \qquad (\alpha = 1, \ldots, 4, 7, \ldots, 10). \qquad \text{(A.2)}$$

Therefore,

$$M_\alpha(x, 5) = \sum_{\substack{j=1 \\ j\neq 5,6}}^{10} M_\alpha(x, j)\left\{\frac{mP_{5j}(1 - mP_{66}) + mP_{56}mP_{6j}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56}mP_{65}}\right\} \tag{A.3}$$

and

$$M_\alpha(x, 6) = \sum_{\substack{j=1 \\ j\neq 5,6}}^{10} M_\alpha(x, j)\left\{\frac{mP_{6j}(1 - mP_{55}) + mP_{65}mP_{5j}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56}mP_{65}}\right\}$$

$(\alpha = 1, \ldots, 4, 7, \ldots, 10)$. (A.4)

So r_{ij}, the (i, j)th element of R, is given by

$$r_{ij} = \frac{-mP_{ij} - mP_{i5} \cdot \gamma_5(j) - mP_{i6} \cdot \gamma_6(j)}{\mu_i}$$

($i = 1, \ldots, 4$, $j = 1, \ldots, 4$, $i \neq j$; for $i = 5, \ldots, 8$ and $j = 5, \ldots, 8$, i and j are replaced on the right-hand side of this equation by $i + 2$ and $j + 2$.)

(A-5)

$$r_{ii} = \frac{1 - mP_{ii} - mP_{i5} \cdot \gamma_5(i) - mP_{i6} \cdot \gamma_6(i)}{\mu_i}$$

($i = 1, \ldots, 4$; for $i = 5, \ldots, 8$, i is replaced by $i + 2$ on the right-hand side),

where

$$\gamma_5(j) = \frac{mP_{5j}(1 - mP_{66}) + mP_{56}mP_{6j}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56}mP_{65}} \tag{A.6}$$

and

$$\gamma_6(j) = \frac{mP_{6j}(1 - mP_{55}) + mP_{65}mP_{5j}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56}mP_{65}} \qquad (j = 1, \ldots, 4, 7, \ldots, 10). \tag{A.7}$$

To calculate exp(Rt) as required in Eq. 26, we use the Jordan decomposition of the matrix R. If m, the mean number of particles produced per collision, is not equal to 1, then the eight eigenvalues of R will in general be distinct; in fact, from a certain symmetry of R, one can show that these eigenvalues form four pairs of numbers with opposite signs. However, if $m = 1$, zero is a double eigenvalue of R; this double eigenvalue causes the Jordan decomposition of R to be nondiagonal. Since this nondiagonality complicates the computations, we exclude the case $m = 1$. If, in fact, numerical results are desired for $m = 1$, we can perform the calculations for $m = 1.0001$ and $m = 0.9999$ and average the results, provided the slab is not nearly critical when $m = 1.0001$. (For $m = 1.0001$ and $m = 0.9999$, there will be two eigenvalues close to zero; experience with the numerical routine JACOBI, used to calculate eigenvalues and eigenvectors, has indicated that it can separate these two nonzero eigenvalues for these m, so that the diagonal Jordan decomposition can be achieved numerically.)

Thus, letting $T = [\underset{\sim}{t}_1 \ldots \underset{\sim}{t}_8]$ be the matrix of eigenvectors of R, we have

$$T^{-1} R \, T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_8 \end{bmatrix}, \tag{A.8}$$

where $\lambda_1, \ldots, \lambda_8$ are the eight distinct eigenvalues of R. The matrices T, T^{-1}, and the numbers $\lambda_1, \ldots, \lambda_8$ may be complex. (It seems that if $m < 1$, all eight eigenvalues, and thus the eigenvectors, are real; however, if $m > 1$, there is always one pair of purely imaginary eigenvalues. There are then six real eigenvectors and two other eigenvectors, complex conjugates of each other, with imaginary parts.)

Since

$$\exp(Rt) = I + \sum_{j=1}^{\infty} (Rt)^j/j!,$$

it follows that

$$T^{-1} \exp(Rt) T = I + T^{-1} R t T + T^{-1} (Rt)^2 T/2! + \ldots .$$

In terms of the matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_8 \end{bmatrix},$$

we can write

$$T^{-1}\exp(Rt)T = I + \Lambda t + (\Lambda t)(\Lambda t)/2! + \ldots = \exp(\Lambda t);$$

therefore,

$$\exp(Rt) = T\exp(\Lambda t)T^{-1}. \tag{A.9}$$

Letting

$$T^{-1} = \begin{bmatrix} \underset{\sim}{r}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{r}_8 \end{bmatrix},$$

we have

$$\exp(Rx) = e^{\lambda_1 x}U_1 + \ldots + e^{\lambda_8 x}U_8, \tag{A.10}$$

where

$$U_i = \underset{\sim}{t}_i \cdot \underset{\sim}{r}_i \quad (i = 1, \ldots, 8). \tag{A.11}$$

Each U_i is an 8 x 8 matrix, since $\underset{\sim}{t}_i$ is a column vector and $\underset{\sim}{r}_i$ is a row vector. The eight matrices U_i are computed.

Now let $M(x)$ be the 8 x 8 matrix

$$\begin{bmatrix} M_1(x,1) & \ldots & M_4(x,1) & M_7(x,1) & \ldots & M_{10}(x,1) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ M_1(x,4) & & & & & \\ M_1(x,7) & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ M_1(x,10) & \ldots & & & & M_{10}(x,10) \end{bmatrix}.$$

Then $M(t) = \exp(Rt) \cdot M(0)$, where $M(0)$ is the 8 x 8 matrix determined by the boundary conditions. This matrix was partitioned as

$$\begin{bmatrix} C_1 & C_2 \\ 0 & I \end{bmatrix},$$

and the solutions for C_1 and C_2 were given after Eq. 29. Thus $M(x) = e^{\lambda_1 x} Z_1 + \dots + e^{\lambda_8 x} Z_8$ $(0 \le x \le t)$, where $Z_i = U_i \cdot M(0)$ $(i = 1, \dots, 8)$. The eight matrices Z_i are calculated. Now we use the fact that $M_\alpha(x, 5)$ and $M_\alpha(x, 6)$ are linear combinations of the $M_\alpha(x, j)$ $(j = 1, \dots, 4, 7, \dots, 10;$ $\alpha = 1, \dots, 4, 7, \dots, 10)$ as given in Eqs. A.3 and A.4. Thus, if we let $M(x)$ be the 10 x 8 matrix

$$\begin{bmatrix} M_1(x, 1) & \dots & M_4(x, 1) & M_7(\lambda, 1) & \dots & M_{10}(x, 1) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ M_1(x, 10) & \dots & M_4(x, 10) & M_7(x, 10) & \dots & M_{10}(x, 10) \end{bmatrix}. \tag{A.12}$$

then $M(x) = e^{\lambda_1 x} W_1 + \dots + e^{\lambda_8 x} W_8$, where the W_i are all 10 x 8 matrices, and

$$\left.\begin{aligned} W_q(i, j) &= Z_q(i, j) \qquad (i = 1, \dots, 4;\ j = 1, \dots, 8), \\ W_q(i+2, j) &= Z_q(i, j) \qquad (i = 5, \dots, 8;\ j = 1, \dots, 8), \\ W_q(5, j) &= \sum_{k=1}^{4} \gamma_5(k) Z_q(k, j) + \sum_{k=5}^{8} \gamma_5(k+2)\, Z_q(k, j) \qquad (j = 1, \dots, 8), \\ W_q(6, j) &= \sum_{k=1}^{4} \gamma_6(k) Z_q(k, j) + \sum_{k=5}^{8} \gamma_6(k+2)\, Z_q(k, j) \qquad (j = 1, \dots, 8) \\ & \qquad (q = 1, \dots, 8), \end{aligned}\right\} \tag{A.13}$$

and $\gamma_5(k)$, $\gamma_6(k)$ are defined by Eqs. A.6 and A.7. The eight matrices W are computed. These matrices completely determine the 10 x 8 matrix $M(x)$, which gives the mean number of particles emergent in each of the eight possible emergent directions for a particle at position x traveling in each of the 10 possible directions of motion. The eight W's, eight U's and eight λ's are all stored for use in the second part of the program.

Now we consider Eq. 30, the basic equation for the second factorial moments:

$$\begin{bmatrix} \mu_1 \frac{\partial}{\partial x} & & 0 \\ & \ddots & \\ 0 & & \mu_{10} \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} S_{\alpha\beta}(x, 1) \\ \vdots \\ S_{\alpha\beta}(x, 10) \end{bmatrix} = \begin{bmatrix} 1-mP_{1,1} & -mP_{1,2} \cdots & -mP_{1,10} \\ \vdots & & \vdots \\ -mP_{10,1} & \cdots & 1-mP_{10,10} \end{bmatrix} \begin{bmatrix} S_{\alpha\beta}(x, 1) \\ \vdots \\ S_{\alpha\beta}(x, 10) \end{bmatrix}$$

$$-m_{(2)} \begin{bmatrix} F_{\alpha\beta}(x, 1) \\ \vdots \\ F_{\alpha\beta}(x, 10) \end{bmatrix}.$$

We recall that

$$F_{\alpha\beta}(x,\ell) = \left\{ \sum_{k=1}^{10} M_\alpha(x, k) P_{\ell k} \right\} \cdot \left\{ \sum_{k=1}^{10} M_\beta(x, k) P_{\ell k} \right\}$$

$(\alpha = 1, \ldots, 4, 7, \ldots, 10;\ \beta = 1, \ldots, 4, 7, \ldots, 10;\ \ell = 1, \ldots, 10)$.

The first factor can be thought of as the ℓth row of the matrix P multiplied by the column vector $\underset{\sim}{M}_\alpha(x)$; writing $M_\alpha(x,\ell)$ in terms of the W's, we get

$$F_{\alpha\beta}(x,\ell) = \left\{ \sum_{k=1}^{10} P_{\ell k} \sum_{q=1}^{8} e^{\lambda_q x} W_q(k, \alpha) \right\} \cdot \left\{ \sum_{m=1}^{10} P_{\ell m} \sum_{r=1}^{8} e^{\lambda_r x} W_r(m, \beta) \right\}$$

$(\alpha = 1, \ldots, 4,\ \beta = 1, \ldots, 4,\ \ell = 1, \ldots, 10;$ for α and $\beta = 7, \ldots, 10$, α and β are replaced by $\alpha - 2$ and $\beta - 2$ on the right-hand side).

This equation can be written

$$F_{\alpha\beta}(x,\ell) = \sum_{q=1}^{8} \sum_{r=1}^{8} e^{(\lambda_q + \lambda_r)x} \cdot X_{\alpha\beta qr}(\ell), \tag{A.14}$$

where

$$X_{\alpha\beta qr}(\ell) = \sum_{k=1}^{10} P_{\ell k} W_q(k,\alpha) \cdot \sum_{m=1}^{10} P_{\ell m} W_r(m,\beta)$$

($\alpha = 1, \ldots, 4$, $\beta = 1, \ldots, 4$, $\ell = 1, \ldots, 10$; for α and $\beta = 7, \ldots, 10$, $\alpha - 2$ and $\beta - 2$ replace α and β on the right side). (A.15)

The X's are calculated.

Next we change the form of the basic equation to that of Eq. 31, i.e.,

$$\frac{\partial}{\partial x}\begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,4) \\ S_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} = R \begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,4) \\ S_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} + \begin{bmatrix} V_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ V_{\alpha\beta}(x,8) \end{bmatrix}.$$

by using the fact that $\mu_5 = \mu_6 = 0$. The 8 x 8 matrix R is the same as in Eq. A.5, and elementary algebra shows that

$$V_{\alpha\beta}(x,\ell) = \left[F_{\alpha\beta}(x,\ell) + \frac{mP_{\ell 5}(1 - mP_{66}) + mP_{\ell 6}mP_{65}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56}mP_{65}} F_{\alpha\beta}(x,5) \right.$$

$$\left. + \frac{mP_{\ell 6}(1 - mP_{55}) + mP_{\ell 5}mP_{56}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56}mP_{65}} F_{\alpha\beta}(x,6)\right]\frac{-m(2)}{\mu_\ell}$$

($\alpha = 1, \ldots, 4$, $\beta = 1, \ldots, 4$, $\ell = 1, \ldots, 4$; similar equations hold for $\alpha = 7, \ldots, 10$, $\beta = 7, \ldots, 10$, and $\ell = 5, \ldots, 8$). (A.16)

This can be simplified to

$$V_{\alpha\beta}(x,\ell) = \sum_{q=1}^{8}\sum_{r=1}^{8} e^{(\lambda_q + \lambda_r)x} \cdot Y_{\alpha\beta qr}(\ell), \qquad (A.17)$$

where

$$Y_{\alpha\beta qr}(\ell) = \left\{ X_{\alpha\beta qr}(\ell) + K_5(\ell) \cdot X_{\alpha\beta qr}(5) + K_6(\ell) \cdot X_{\alpha\beta qr}(6) \right\} \frac{-m_{(2)}}{\mu_\ell}$$

$(\alpha = 1, \ldots, 4, \ldots, 7, \ldots, 10,\ \beta = 1, \ldots, 4, \ldots, 7, \ldots, 10,\ \ell = 1, \ldots, 4;$
for $\ell = 5, \ldots, 8$, the ℓ in X and in μ_ℓ is replaced by $\ell + 2$) (A.18)

and

$$K_5(\ell) = \frac{mP_{\ell 5}(1 - mP_{66}) + mP_{\ell 6} mP_{65}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56} mP_{65}}$$

($\ell = 1, \ldots, 4$; for $\ell = 5, \ldots, 8$, ℓ is replaced by $\ell + 2$), (A.19)

$$K_6(\ell) = \frac{mP_{\ell 6}(1 - mP_{55}) + mP_{\ell 5} mP_{56}}{(1 - mP_{55})(1 - mP_{66}) - mP_{56} P_{65}}$$

($\ell = 1, \ldots, 4$; for $\ell = 5, \ldots, 8$, ℓ is replaced by $\ell + 2$). (A.20)

The K's and the Y's are computed. Now the solution is

$$\begin{bmatrix} S_{\alpha\beta}(x, 1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x, 4) \\ S_{\alpha\beta}(x, 7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x, 10) \end{bmatrix} = \exp(Rx) \left\{ \underset{\sim}{c}_{\alpha\beta} + \int_0^x \exp(-yR) \sum_{q=1}^{8} \sum_{r=1}^{8} \underset{\sim}{Y}_{\alpha\beta qr} \cdot e^{(\lambda_q + \lambda_r) y}\, dy \right\}. \tag{A.21}$$

Next we write $\exp(-yR)$ as

$$\sum_{s=1}^{8} e^{-\lambda_s y} U_s .$$

Then the integral in Eq. A.21 can be written as

$$\sum_{s=1}^{8} \sum_{q=1}^{8} \sum_{r=1}^{8} U_s \cdot \underset{\sim}{Y}_{\alpha\beta qr} \cdot \int_0^x e^{(\lambda_q + \lambda_r - \lambda_s) y}\, dy. \tag{A.22}$$

Now the reason for breaking up

$$\int_0^x \exp(-yR) \cdot \underset{\sim}{Y}_{\alpha\beta}(y)\, dy$$

into the triple summation of Eq. A.22 is apparent, since

$$\int_0^x e^{(\lambda_q + \lambda_r - \lambda_s)\, y}\, dy$$

can be integrated analytically. Therefore the solution is

$$\begin{bmatrix} S_{\alpha\beta}(x,1) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,4) \\ S_{\alpha\beta}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{\alpha\beta}(x,10) \end{bmatrix} = \exp(xR)\left\{ \underset{\sim}{c}_{\alpha\beta} + \sum_{s=1}^{8} \sum_{q=1}^{8} \sum_{r=1}^{8} U_s \cdot \underset{\sim}{Y}_{\alpha\beta qr} \cdot \left[\frac{e^{(\lambda_q + \lambda_r - \lambda_s)x} - 1}{\lambda_q + \lambda_r - \lambda_s}\right]\right\}. \quad (A.23)$$

assuming $\lambda_q + \lambda_r - \lambda_s \neq 0$ $(s = 1, \ldots, 8;\ q = 1, \ldots, 8;\ r = 1, \ldots, 8)$; the eight-component vector $\underset{\sim}{c}_{\alpha\beta}$ is determined by the boundary conditions. Since $\underset{\sim}{S}_{\alpha\beta}(0) = \underset{\sim}{c}_{\alpha\beta}$, the bottom four components of $\underset{\sim}{c}_{\alpha\beta}$ are zeros.

The symmetry of directions 1, ..., 4 with directions 10, ..., 7 shows that the $S_{\alpha\beta}$'s need only be calculated for $\alpha = 1, \ldots, 4$ and $\beta = 1, \ldots, 4, 7, \ldots, 10$; the symmetry relations are $S_{\alpha\beta}(x, \ell) = S_{11-\alpha,11-\beta}(t - x, 11 - \ell)$ and $S_{\alpha\beta}(x,\ell) = S_{\beta\alpha}(x,\ell)$ $(\alpha = 1, \ldots, 4, 7, \ldots, 10;\ \beta = 1, \ldots, 4, 7, \ldots, 10;\ \ell = 1, \ldots, 4;\ 0 \le x \le t)$. Thus a complete solution is obtained by evaluating $S_{\alpha\beta}(x, \ell)$ for $\alpha = 1, \ldots, 4$, $\beta = 1, \ldots, 4, 7, \ldots, 10$, and $\ell = 1, \ldots, 4, 7, \ldots, 10$ $(0 \le x \le t)$. Let $S(x)$ be the 8 x 32 matrix

$$\begin{bmatrix} S_{11}(x,1) \ldots S_{14}(x,1) S_{17}(x,1) \ldots S_{1,10}(x,1) S_{21}(x,1) \ldots S_{2,10}(x,1) S_{31}(x,1) \ldots S_{4,10}(x,1) \\ \cdot \qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ S_{11}(x,4) \\ S_{11}(x,7) \\ \cdot \\ \cdot \\ \cdot \\ S_{11}(x,10) \ldots \qquad\qquad\qquad\qquad S_{4,10}(x,10) \end{bmatrix}. \quad (A.24)$$

and let $\begin{bmatrix} \mathrm{Int1}(x) \\ \mathrm{Int2}(x) \end{bmatrix}$ be the 8 x 32 matrix

$$[\underset{\sim}{\mathrm{Int}}_{1,1}(x)\underset{\sim}{\mathrm{Int}}_{1,2}(x)\ldots\underset{\sim}{\mathrm{Int}}_{1,4}(x)\underset{\sim}{\mathrm{Int}}_{1,7}(x)\ldots\underset{\sim}{\mathrm{Int}}_{1,10}(x)\underset{\sim}{\mathrm{Int}}_{2,1}(x)\ldots\underset{\sim}{\mathrm{Int}}_{4,10}(x)],$$

where $\underset{\sim}{\mathrm{Int}}_{\alpha\beta}(x)$ is the eight-component column vector

$$\sum_{s=1}^{8}\sum_{q=1}^{8}\sum_{r=1}^{8} U_s \cdot \underset{\sim}{Y}_{\alpha\beta qr} \cdot \left[\frac{e^{(\lambda_q+\lambda_r-\lambda_s)x} - 1}{\lambda_q+\lambda_r-\lambda_s}\right]. \tag{A.25}$$

Thus Int1(x) and Int2(x) are each 4 x 32 matrices. Finally, let C be the 4 x 32 matrix

$$[\underset{\sim}{c}_{11}\cdots\underset{\sim}{c}_{14}\underset{\sim}{c}_{17}\cdots\underset{\sim}{c}_{1,10}\underset{\sim}{c}_{21}\cdots\underset{\sim}{c}_{4,10}],$$

where the $\underset{\sim}{c}_{\alpha\beta}$'s are the top four (i.e., nonzero) components of the $\underset{\sim}{c}_{\alpha\beta}$'s used in Eq. A.23.

Then we have the equation

$$S(x) = \exp(Rx)\left\{\begin{bmatrix} C \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{Int1}(x) \\ \mathrm{Int2}(x) \end{bmatrix}\right\}. \tag{A.26}$$

To compute C, we use the boundary conditions on $S(t)$; i.e., $S_{\alpha\beta}(t,\ell) = 0$ ($\alpha = 1, \ldots, 4$, $\beta = 1, \ldots, 4, 7, \ldots, 10$, $\ell = 1, \ldots, 4$). Thus we compute

$$\begin{bmatrix} \mathrm{Int1}(t) \\ \mathrm{Int2}(t) \end{bmatrix}$$

using Eq. A.25. Then letting

$$\exp(Rt = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array}\right],$$

we have

$$0 = E_{11} \cdot [C + \mathrm{Int1}(t)] + E_{12}[\mathrm{Int2}(t)]. \tag{A.27}$$

Therefore, $C = -E_{11}^{-1}E_{12}\,\mathrm{Int2}(t) - \mathrm{Int1}(t)$. The 4 x 32 matrix C is computed using this formula. Then $S(x)$ can be calculated for arbitrary x ($0 \le x \le t$) using Eqs. A.26 and A.25.

The transformations from the second factorial moments S to the more meaningful variances, covariances, correlations, and standard deviations are then made using the standard formulas.

It has been observed that the method of calculation becomes numerically unstable when t gets very close to the critical length t_c ($t_c - t \approx 0.001$). A similar phenomenon occurred in the nonmultiplicative case (Brockwell[3]) when t becomes large (approximately seven mean free paths). However, in the nonmultiplicative case, the range of accuracy was increased considerably by considering thick slabs to be superpositions of thinner slabs (see Ref. 3, Chapter VI, Section 5). An analogous approach could be used to study thick slabs in the multiplicative case, but so far this approach has not been investigated.

APPENDIX B

Listing of Computer Program

```
LEVEL 15 ( 1 JAN 68)                         OS/360  FORTRAN H  DATE  68.261/22.11.32

      COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=57,SOURCE,EBCDIC,NOLIST,NODECK,
                                                          LOAD,MAP,NOEDIT,ID,NOXREF
               C
               C      STOCHASTIC TRANSPORT PROGRAM II
               C
ISN 0002              IMPLICIT REAL*8(A-H,O-Z)
ISN 0003              REAL*8 LENGTH,M,M2,MU,KAPPA5,KAPPA6
ISN 0004              COMPLEX*16   COMP,  CR,  CT,  CTEMP,  CTINV,  D,  DETERM,  E11,
                     1  E12,  FACTOR,  INT,  INT1,  INT2,  MEAN,  SCONST,  SMOP,  THETA,
                     2  U,  VTEMP,  W,  XTEMP1,  XTEMP2,  XX,  Y,  Z
ISN 0005              DIMENSION   B2(4),  COMP(10,10),  CONST(8,8),  CORR(8,8),
                     1  CR(8,8),  CT(8,8),  CTEMP(8,8),  CTINV(8,8),  D(4,4),  E11(4,4),
                     2  E12(4,4),  GAM5(10),  GAM6(10),  INT(8,32),  INT1(4,32),
                     3  INT2(4,32),  KAPPA5(10),  KAPPA6(10),  MEAN(10,8),  MU(10),
                     4  OCOV(8,32),  OMEAN(10,8),  P(10,10),  R(8,8),  RR(8,8),
                     5  RSC(8,32),  SCAT(9),  SCONST(8,32),  SDEV(8),  SML(4,32),
                     6  SMOP(8,32),  T(8,8),  THETA(8),  U(8,8,8),  VAR(4,8),  VTEMP(8),
                     7  W(8,10,8),  XX(4,8,8,8,10),  Y(4,8,8,8,8),  Z(8,8,8)
ISN 0006              FIVE = 5.0
ISN 0007              MU(1) = 1.0
ISN 0008              MU(2) = (DSQRT(FIVE) + 1.0)/4.0
ISN 0009              MU(3) = 0.5
ISN 0010              MU(4) = (DSQRT(FIVE) - 1.0)/4.0
ISN 0011              MU(5) = 0.0
ISN 0012              DO 102 I = 1,5
ISN 0013          102 MU(11-I) = -MU(I)
ISN 0014              READ (5,902) ISCAT
ISN 0015              IF (ISCAT) 103,106,108
               C
               C      IF ISCAT = 0, ISOTROPIC SCATTERING;  IF ISCAT = -1, RAYLEIGH
               C      SCATTERING;  IF ISCAT = +1, A DIFFERENT SCATTERING LAW IS READ IN.
               C
ISN 0016          103 SCAT(1) = 0.050
ISN 0017              SCAT(2) = (11.0 + DSQRT(FIVE))/320.0
ISN 0018              SCAT(3) = 1.0/32.0
ISN 0019              SCAT(4) = (11.0 - DSQRT(FIVE))/320.0
ISN 0020              SCAT(5) = 0.0250
ISN 0021              DO 104 I = 6,9
ISN 0022          104 SCAT(I) = SCAT(10-I)
ISN 0023              WRITE (6,835)
ISN 0024              GO TO 109
ISN 0025          106 DO 107 I=1,9
ISN 0026          107 SCAT(I) = 1.0/30.0
ISN 0027              WRITE (6,837)
ISN 0028              GO TO 109
ISN 0029          108 READ (5,900) (SCAT(I),I=1,9)
ISN 0030          109 CONTINUE
               C
               C      COMPUTATION OF MATRIX P
               C
ISN 0031              P(1,1) = SCAT(1)
ISN 0032              P(1,2) = 4.0*SCAT(2)
ISN 0033              P(1,3) = 4.0*SCAT(3)
ISN 0034              P(1,4) = 4.0*SCAT(4)
ISN 0035              P(2,1) = SCAT(2)
ISN 0036              P(2,2) = SCAT(1) + SCAT(2) + SCAT(3) + SCAT(4)
ISN 0037              P(2,3) = SCAT(2) + SCAT(3) + SCAT(4) + SCAT(5)
ISN 0038              P(2,4) = SCAT(2) + SCAT(3) + SCAT(5) + SCAT(6)
ISN 0039              P(3,1) = SCAT(3)
ISN 0040              P(3,2) = P(2,3)
ISN 0041              P(3,3) = SCAT(1) + SCAT(2) + SCAT(6) + SCAT(7)
ISN 0042              P(3,4) = SCAT(2) + SCAT(4) + SCAT(5) + SCAT(7)
ISN 0043              P(4,1) = SCAT(4)
ISN 0044              P(4,2) = P(2,4)
ISN 0045              P(4,3) = P(3,4)
```

```
ISN 0046          P(4,4) = SCAT(1) + SCAT(3) + SCAT(6) + SCAT(8)
ISN 0047          DO 110 I=7,10
ISN 0048          DO 110 J=7,10
ISN 0049      110 P(I,J) = P(11-I,11-J)
ISN 0050          P(7,1) = SCAT(6)
ISN 0051          P(7,2) = SCAT(4) + SCAT(5) + SCAT(7) + SCAT(8)
ISN 0052          P(7,3) = SCAT(3) + SCAT(5) + SCAT(6) + SCAT(8)
ISN 0053          P(7,4) = SCAT(2) + SCAT(4) + SCAT(7) + SCAT(9)
ISN 0054          P(8,1) = SCAT(7)
ISN 0055          P(8,2) = SCAT(5) + SCAT(6) + SCAT(7) + SCAT(8)
ISN 0056          P(8,3) = SCAT(3) + SCAT(8) + SCAT(4) + SCAT(9)
ISN 0057          P(8,4) = SCAT(3) + SCAT(5) + SCAT(6) + SCAT(8)
ISN 0058          P(9,1) = SCAT(8)
ISN 0059          P(9,2) = SCAT(6) + SCAT(7) + SCAT(8) + SCAT(9)
ISN 0060          P(9,3) = SCAT(5) + SCAT(6) + SCAT(7) + SCAT(8)
ISN 0061          P(9,4) = SCAT(4) + SCAT(5) + SCAT(7) + SCAT(8)
ISN 0062          P(10,1) = SCAT(9)
ISN 0063          P(10,2) = 4.0*SCAT(8)
ISN 0064          P(10,3) = 4.0*SCAT(7)
ISN 0065          P(10,4) = 4.0*SCAT(6)
ISN 0066          DO 115 I=1,4
ISN 0067          DO 115 J=7,10
ISN 0068      115 P(I,J) = P(11-I,11-J)
ISN 0069          P(5,1) = SCAT(5)
ISN 0070          P(5,2) = 2.0*SCAT(4) + 2.0*SCAT(6)
ISN 0071          P(5,3) = 2.0*SCAT(2) + 2.0*SCAT(8)
ISN 0072          P(5,4) = 2.0*SCAT(3) + 2.0*SCAT(7)
ISN 0073          P(6,1) = SCAT(5)
ISN 0074          P(6,2) = 2.0*SCAT(3) + 2.0*SCAT(7)
ISN 0075          P(6,3) = 2.0*SCAT(4) + 2.0*SCAT(6)
ISN 0076          P(6,4) = 2.0*SCAT(2) + 2.0*SCAT(8)
ISN 0077          DO 120 I=5,6
ISN 0078          DO 120 J=7,10
ISN 0079      120 P(I,J) = P(I,11-J)
ISN 0080          P(1,5) = 2.0*SCAT(5)
ISN 0081          P(1,6) = P(1,5)
ISN 0082          P(2,5) = SCAT(4) + SCAT(6)
ISN 0083          P(2,6) = SCAT(3) + SCAT(7)
ISN 0084          P(3,5) = SCAT(2) + SCAT(8)
ISN 0085          P(3,6) = SCAT(4) + SCAT(6)
ISN 0086          P(4,5) = SCAT(3) + SCAT(7)
ISN 0087          P(4,6) = SCAT(2) + SCAT(8)
ISN 0088          P(5,5) = SCAT(1) + SCAT(9)
ISN 0089          P(5,6) = 2.0*SCAT(5)
ISN 0090          P(6,5) = P(5,6)
ISN 0091          P(6,6) = P(5,5)
ISN 0092          DO 125 I=7,10
ISN 0093          DO 125 J=5,6
ISN 0094      125 P(I,J) = P(11-I,J)
ISN 0095          WRITE (6,905)
ISN 0096          WRITE (6,910) ((P(I,J),J=1,10),I=1,10)
            C
            C     READ IN M,M2, AND NUMBER OF LENGTHS FOR THIS MEAN
            C
ISN 0097      130 READ (5,880) M,M2,NUMLTH
ISN 0098          IF (M .LE. 0.0) GO TO 500
            C
            C     COMPUTATION OF MATRIX R
            C
ISN 0100          DEN = (M*P(5,5) - 1.0) *(M*P(6,6) - 1.0) - (M*P(6,5))*(M*P(5,6))
ISN 0101          DO 140 I=1,4
ISN 0102          GAM6(I) = ((M*P(6,5))*(M*P(5,I)) - (M*P(6,I))*(M*P(5,5)-1.0))/DEN
ISN 0103          GAM6(I+6) = ((M*P(6,5))*(M*P(5,I+6)) - (M*P(6,I+6))*(M*P(5,5)-1.0)
                 1)/DEN
ISN 0104          GAM5(I) = ((M*P(5,6))*(M*P(6,I)) - (M*P(5,I))*(M*P(6,6)-1.0))/DEN
ISN 0105          GAM5(I+6) =  (M*P(5,6)*(M*P(6,I+6)) - (M*P(5,I+6))*(M*P(6,6)-1.0))
                 1/DEN
ISN 0106      140 CONTINUE
ISN 0107          DO 150 I=1,4
ISN 0108          DO 150 J=1,4
ISN 0109          R(I,J) = (-M*P(I,J) - M*P(I,5)*GAM5(J) - M*P(I,6)*GAM6(J))/MU(I)
```

```
                      R(I,J+4) = (-M*P(I,J+6) - M*P(I,5)*GAM5(J+6) - M*P(I,6)*GAM6(J+6))
ISN 0110             1/MU(I)
ISN 0111              R(I+4,J) = (-M*P(I+6,J) - M*P(I+6,5)*GAM5(J) -M*P(I+6,6)*GAM6(J))/
                     1MU(I+6)
ISN 0112              R(I+4,J+4) = (-M*P(I+6,J+6) - M*P(I+6,5)*GAM5(J+6)-M*P(I+6,6)*GAM6
                     1(J+6))/MU(I+6)
ISN 0113          150 CONTINUE
ISN 0114              DO 155 I= 1,4
ISN 0115              R(I,I) = R(I,I) + (1.0/MU(I))
ISN 0116              R(I+4,I+4) = R(I+4,I+4) + (1.0/MU(I+6))
ISN 0117          155 CONTINUE
ISN 0118              WRITE (6,915)
ISN 0119              WRITE (6,975) ((R(I,J),J=1,8),I=1,8)
ISN 0120              DO 170 I=1,8
ISN 0121              DO 170 J=1,8
ISN 0122          170 RR(I,J) = R(I,J)
                C
                C       COMPUTATION OF EIGENVALUES & EIGENVECTORS
                C
ISN 0123              KONST = 1
ISN 0124              CALL CLOCK(T1)
ISN 0125              CALL JACOBI (RR,T,8,KONST,8)
ISN 0126              CALL CLOCK(T2)
ISN 0127              TT = T2 - T1
ISN 0128              WRITE (6,945) KONST,TT
ISN 0129              WRITE (6,950)
ISN 0130              WRITE (6,965) ((RR(I,J),J=1,8),I=1,8)
ISN 0131              WRITE (6,960)
ISN 0132              WRITE (6,965) ((T(I,J),J=1,8),I=1,8)
ISN 0133              IDUM = 0
ISN 0134              DO 180 I=1,7
ISN 0135              IF (IDUM .NE.1) GO TO 171
ISN 0137              IDUM = 0
ISN 0138              GO TO 180
ISN 0139          171 IF (DABS(RR(I,I+1)) .GE. 1.0D-07) GO TO 174
ISN 0141              THETA (I) = RR(I,I)
ISN 0142              DO 172 J=1,8
ISN 0143          172 CT(J,I) = T(J,I)
ISN 0144              GO TO 180
ISN 0145          174 IF (DABS(RR(I,I)) .LE. 1.0D-07) GO TO 175
ISN 0147              THETA(I) = RR(I,I)
ISN 0148              THETA(I) = THETA(I) + (0.0,1.0)*RR(I,I+1)
ISN 0149              THETA(I+1) = RR(I+1,I+1)
ISN 0150              THETA(I+1) = THETA(I+1) + (0.0,1.0)*RR(I+1,I)
ISN 0151              GO TO 176
ISN 0152          175 THETA(I) = RR(I,I+1)*(0.0,1.0)
ISN 0153              THETA(I+1) = RR(I+1,I)*(0.0,1.0)
ISN 0154          176 DO 178 J=1,8
ISN 0155              CT(J,I) = T(J,I)
ISN 0156              CT(J,I) = CT(J,I) + (0.0,1.0)*T(J,I+1)
ISN 0157              CT(J,I+1) = T(J,I)
ISN 0158          178 CT(J,I+1) = CT(J,I+1) - (0.0,1.0)*T(J,I+1)
ISN 0159              IDUM = 1
ISN 0160          180 CONTINUE
ISN 0161              IF (DABS(RR(7,8)) .GE. 1.0D-07) GO TO 183
ISN 0163              THETA(8) = RR(8,8)
ISN 0164              DO 182 J=1,8
ISN 0165          182 CT(J,8) = T(J,8)
ISN 0166          183 DO 184 I=1,8
ISN 0167              DO 184 J=1,8
ISN 0168          184 CTINV(I,J) = CT(I,J)
ISN 0169              WRITE (6,955)
ISN 0170              WRITE (6,908) (THETA(I),I=1,8)
                C
                C       CHECK OF DIAGONALIZATION, AND COMPUTATION OF THE 8 U'S
                C
ISN 0171              CALL MATINV(CTINV,8,SDEV,0,DETERM,8)
ISN 0172              WRITE (6,960)
ISN 0173              WRITE (6,964) ((CT(I,J),J=1,8),I=1,8)
ISN 0174              WRITE (6,970)
ISN 0175              WRITE (6,964) ((CTINV(I,J),J=1,8),I=1,8)
```

```
ISN 0176              DO 185 I=1,8
ISN 0177              DO 185 J=1,8
ISN 0178          185 CR(I,J) = R(I,J)
ISN 0179              CALL CMTMLT (CTINV,8,8,CR,8,8,CTEMP)
ISN 0180              CALL CMTMLT (CTEMP,8,8,CT,8,8,CR)
ISN 0181              WRITE (6,920)
ISN 0182              WRITE (6,964) ((CR(I,J),J=1,8),I=1,8)
ISN 0183              DO 187 I=1,8
ISN 0184              DO 187 J=1,8
ISN 0185              DO 187 K=1,8
ISN 0186          187 U(I,J,K) = CT(J,I)*CTINV(I,K)
                C
                C     READ IN (NUMLTH) LENGTHS FOR THIS MEAN
                C
ISN 0187              KCOUNT = 0
ISN 0188          190 READ (5,900) LENGTH
ISN 0189              KCOUNT = KCOUNT + 1
ISN 0190              WRITE (6,918) LENGTH
                C
                C     CALCULATION OF EXP(R*LENGTH)
                C
ISN 0191              DO 195 I=1,8
ISN 0192              DO 195 J=1,8
ISN 0193          195 CR(I,J) = 0.0
ISN 0194              DO 200 K=1,8
ISN 0195              FACTOR = CDEXP(THETA(K)*LENGTH)
ISN 0196              DO 200 I=1,8
ISN 0197              DO 200 J=1,8
ISN 0198          200 CR(I,J) = CR(I,J) + U(K,I,J)*FACTOR
ISN 0199              WRITE (6,985)
ISN 0200              WRITE (6,964) ((CR(I,J),J=1,8),I=1,8)
                C
                C     CALCULATION OF EXP(R*LENGTH) USING H. G. SUBROUTINE
                C
ISN 0201              CALL EXMATR(R,T,LENGTH,8,CORR,CONST,SDEV)
ISN 0202              WRITE (6,986)
ISN 0203              WRITE (6,975) ((T(I,J), J=1,8),I=1,8)
                C
                C     CALCULATION OF CONSTANTS MATRIX
                C
ISN 0204          209 DO 210 I=1,4
ISN 0205              DO 210 J=1,4
ISN 0206              REAL = CR(I,J)
ISN 0207              E11(I,J) = REAL
ISN 0208              REAL = CR(I,J+4)
ISN 0209              E12(I,J) = REAL
ISN 0210              CONST(I+4,J) = 0.0
ISN 0211          210 CONST(I+4,J+4) = 0.0
ISN 0212              DO 215 I=5,8
ISN 0213          215 CONST (I,I) = 1.0
ISN 0214              CALL MATINV (E11,4,B2,0,DETERM,4)
ISN 0215              WRITE (6,946) DETERM
ISN 0216              WRITE (6,964) ((E11(I,J),J=1,4),I=1,4)
ISN 0217              DO 220 I=1,4
ISN 0218              DO 220 J=1,4
ISN 0219          220 CONST (I,J) = E11(I,J)
ISN 0220              CALL CMTMLT (E11,4,4,E12,4,4,D)
ISN 0221              DO 225 I=1,4
ISN 0222              DO 225 J=1,4
ISN 0223          225 CONST (I,J+4) = -D(I,J)
ISN 0224              WRITE (6,990)
ISN 0225              WRITE (6,965) ((CONST(I,J),J=1,8),I=1,8)
                C
                C      CALCULATION OF MATRICES Z'S AND W'S
                C
ISN 0226              DO 240 I=1,8
ISN 0227              DO 240 J=1,8
ISN 0228              DO 240 K=1,8
ISN 0229              Z(I,J,K) = 0.0
ISN 0230              DO 240 L=1,8
ISN 0231          240 Z(I,J,K) = Z(I,J,K) + U(I,J,L)*CONST(L,K)
```

```
ISN 0232          DO 275 N1=1,8
ISN 0233          DO 270 J=1,8
ISN 0234          DO 250 I=1,4
ISN 0235      250 W(N1,I,J) = Z(N1,I,J)
ISN 0236          DO 255 I=5,8
ISN 0237      255 W(N1,I+2,J) = Z(N1,I,J)
ISN 0238          W(N1,5,J) = 0.0
ISN 0239          W(N1,6,J) = 0.0
ISN 0240          DO 260 L=1,4
ISN 0241          W(N1,5,J) = W(N1,5,J) + GAM5(L)*Z(N1,L,J)
ISN 0242      260 W(N1,6,J) = W(N1,6,J) + GAM6(L)*Z(N1,L,J)
ISN 0243          DO 265 L=5,8
ISN 0244          W(N1,5,J) = W(N1,5,J) + GAM5(L+2)*Z(N1,L,J)
ISN 0245      265 W(N1,6,J) = W(N1,6,J) + GAM6(L+2)*Z(N1,L,J)
ISN 0246      270 CONTINUE
ISN 0247      275 CONTINUE
ISN 0248      280 READ (5,930) X
ISN 0249          IF (X) 285,290,290
            C
            C     CALCULATION OF MEAN MATRIX
            C
ISN 0250      290 DO 292 I=1,10
ISN 0251          DO 292 J=1,8
ISN 0252          MEAN(I,J) = 0.0
ISN 0253          DO 292 K=1,8
ISN 0254      292 MEAN(I,J) = MEAN(I,J) + (CDEXP(THETA(K)*X))*W(K,I,J)
ISN 0255          WRITE (6,935) LENGTH,X
ISN 0256          WRITE (6,937) M
ISN 0257          DO 293 I=1,10
ISN 0258          DO 293 J=1,8
ISN 0259      293 OMEAN(I,J) = MEAN(I,J)
ISN 0260          WRITE (6,940)
ISN 0261          WRITE (6,965) ((OMEAN(I,J), J=1,8),I=1,10)
ISN 0262      296 DO 297 I=1,10
ISN 0263          DO 297 J=1,8
ISN 0264      297 OMEAN(I,J) = (0.0,-1.0)*MEAN(I,J)
ISN 0265          WRITE (6,967)
ISN 0266          WRITE (6,965) ((OMEAN(I,J), J=1,8),I=1,10)
ISN 0267          GO TO 280
            C
            C     CALCULATION OF SECOND FACTORIAL MOMENTS
            C     FIRST, CALCULATION OF X'S
            C
ISN 0268      285 CONTINUE
ISN 0269          STDEV = DSQRT(M2 + M - M*M)
ISN 0270          WRITE (6,840) M,STDEV
ISN 0271          DO 308 I = 1,10
ISN 0272          DO 308 J = 1,10
ISN 0273      308 COMP(I,J) = P(I,J)
ISN 0274          DO 310 IALPHA = 1,4
ISN 0275          DO 310 IBETA = 1,8
ISN 0276          DO 310 IQ = 1,8
ISN 0277          DO 310 IR = 1,8
ISN 0278          DO 310 IL = 1,10
ISN 0279          XTEMP1 = 0.0
ISN 0280          XTEMP2 = 0.0
ISN 0281          DO 305 IK = 1,10
ISN 0282          XTEMP1 = XTEMP1 + COMP(IL,IK)*W(IQ,IK,IALPHA)
ISN 0283      305 XTEMP2 = XTEMP2 + COMP(IL,IK)*W(IR,IK,IBETA)
ISN 0284      310 XX(IALPHA,IBETA,IQ,IR,IL) = XTEMP1*XTEMP2
ISN 0285          DO 320 L=1,4
ISN 0286          KAPPA5(L) = (M*P(L,5)*(1.0-M*P(6,6)) + M*P(L,6)*M*P(6,5))/DEN
ISN 0287      320 KAPPA6(L) = (M*P(L,5)*M*P(5,6) + M*P(L,6)*(1.0-M*P(5,5)))/DEN
ISN 0288          DO 325 L = 5,8
ISN 0289          KAPPA5(L) = (M*P(L+2,5)*(1.0 - M*P(6,6)) + M*P(L+2,6)*M*P(6,5))/DE
                 1N
ISN 0290      325 KAPPA6(L) = (M*P(L+2,5)*M*P(5,6) + M*P(L+2,6)*(1.0-M*P(5,5)))/DEN
            C
            C     CALCULATION OF Y'S
            C
ISN 0291          RMM2 = -M2
```

```
ISN 0292          DO 335 IALPHA = 1,4
ISN 0293          DO 335 IBETA = 1,8
ISN 0294          DO 335 IQ = 1,8
ISN 0295          DO 335 IR = 1,8
ISN 0296          DO 335 IL = 1,4
ISN 0297          Y(IALPHA,IBETA,IQ,IR,IL) = (XX(IALPHA,IBETA,IQ,IR,IL) + KAPPA5(IL)
                 1*XX(IALPHA,IBETA,IQ,IR,5) + KAPPA6(IL)*XX(IALPHA,IBETA,IQ,IR,6))*
                 2(RMM2/MU(IL))
ISN 0298      335 Y(IALPHA,IBETA,IQ,IR,IL+4) = (XX(IALPHA,IBETA,IQ,IR,IL+6) + KAPPA5
                 1(IL + 4)*XX(IALPHA,IBETA,IQ,IR,5) + KAPPA6(IL + 4)*XX(IALPHA,IBETA
                 2,IQ,IR,6))*(RMM2/MU(IL+6))
            C
            C     CALCULATION OF INTEGRALS
            C
ISN 0299          DO 345 IALPHA = 1,4
ISN 0300          DO 345 IBETA = 1,8
ISN 0301          DO 336 IM = 1,8
ISN 0302      336 INT(IM,8*(IALPHA-1) + IBETA) = 0.0
ISN 0303          DO 345 IS = 1,8
ISN 0304          DO 337 IM = 1,8
ISN 0305      337 VTEMP(IM) = 0.0
ISN 0306          DO 340 IQ = 1,8
ISN 0307          DO 340 IR = 1,8
ISN 0308          FACTOR = (CDEXP((THETA(IQ) + THETA(IR) - THETA(IS))*LENGTH) - 1.0)
                 1/(THETA(IQ) + THETA(IR) - THETA(IS))
ISN 0309          DO 340 IM = 1,8
ISN 0310      340 VTEMP(IM) = VTEMP(IM) + Y(IALPHA,IBETA,IQ,IR,IM)*FACTOR
ISN 0311          DO 344 IL = 1,8
ISN 0312          DO 344 IM = 1,8
ISN 0313      344 INT(IL,8*(IALPHA-1) + IBETA) = INT(IL,8*(IALPHA-1) + IBETA) +
                 1U(IS,IL,IM)*VTEMP(IM)
ISN 0314      345 CONTINUE
            C
            C     CALCULATION OF CONSTANTS MATRIX
            C
ISN 0315          DO 347 I = 1,4
ISN 0316          DO 347 J = 1,32
ISN 0317      347 INT2(I,J) = INT(I + 4,J)
ISN 0318          DO 350 I = 1,8
ISN 0319          DO 350 J = 1,8
ISN 0320      350 CR(I,J) = 0.0
ISN 0321          DO 352 K = 1,8
ISN 0322          FACTOR = CDEXP(THETA(K)*LENGTH)
ISN 0323          DO 352 I = 1,8
ISN 0324          DO 352 J = 1,8
ISN 0325      352 CR(I,J) = CR(I,J) + U(K,I,J)*FACTOR
ISN 0326          DO 355 I = 1,4
ISN 0327          DO 355 J = 1,4
ISN 0328          E11(I,J) = CR(I,J)
ISN 0329      355 E12(I,J) = CR(I,J+4)
ISN 0330          CALL CMTMLT (E12,4,4,INT2,4,32,INT1)
ISN 0331          CALL MATINV (E11,4,B2,0,DETERM,4)
ISN 0332          CALL CMTMLT ( E11,4,4,INT1,4,32,INT2)
ISN 0333          DO 360 I = 1,4
ISN 0334          DO 360 J = 1,32
ISN 0335          SCONST(I+4,J) = 0.0
ISN 0336      360 SCONST(I,J) = -INT2(I,J) - INT(I,J)
ISN 0337          DO 362 I = 1,8
ISN 0338          DO 362 J = 1,32
ISN 0339      362 RSC(I,J) = SCONST(I,J)
ISN 0340          WRITE (6,1005)
ISN 0341          WRITE (6,1010) ((RSC(I,J), J=1,32),I=1,8)
ISN 0342          DO 365 I=1,8
ISN 0343          DO 365 J=1,32
ISN 0344      365 OCOV(I,J) = (0.0,-1.0)*SCONST(I,J)
ISN 0345          WRITE (6,1020)
ISN 0346          WRITE (6,1010) ((OCOV(I,J), J=1,32),I=1,8)
            C
            C     CALCULATIONS FOR ARBITRARY X
            C
ISN 0347      370 READ (5,930) X
```

```
ISN 0348              IF (X) 450,375,375
ISN 0349          375 DO 385 IALPHA = 1,4
ISN 0350              DO 385 IBETA = 1,8
ISN 0351              DO 376 IM = 1,8
ISN 0352          376 INT(IM,8*(IALPHA-1) + IBETA) = 0.0
ISN 0353              DO 385 IS = 1,8
ISN 0354              DO 377 IM = 1,8
ISN 0355          377 VTEMP(IM) = 0.0
ISN 0356              DO 380 IQ = 1,8
ISN 0357              DO 380 IR = 1,8
ISN 0358              FACTOR = (CDEXP((THETA(IQ) + THETA(IR) - THETA(IS))*X) - 1.0)/(THE
                     1TA(IQ) + THETA(IR) - THETA(IS))
ISN 0359              DO 380 IM = 1,8
ISN 0360          380 VTEMP(IM) = VTEMP(IM) + Y(IALPHA,IBETA,IQ,IR,IM)*FACTOR
ISN 0361              DO 384 IL = 1,8
ISN 0362              DO 384 IM = 1,8
ISN 0363          384 INT(IL,8*(IALPHA-1) + IBETA) = INT(IL,8*(IALPHA-1) + IBETA) +
                     1U(IS,IL,IM)*VTEMP(IM)
ISN 0364          385 CONTINUE
ISN 0365              DO 390 I = 1,4
ISN 0366              DO 390 J = 1,32
ISN 0367          390 INT(I,J) = INT(I,J) + SCONST(I,J)
ISN 0368              DO 400 I = 1,8
ISN 0369              DO 400 J = 1,8
ISN 0370          400 CR(I,J) = 0.0
ISN 0371              DO 405 K = 1,8
ISN 0372              FACTOR = CDEXP(THETA(K)*X)
ISN 0373              DO 405 I = 1,8
ISN 0374              DO 405 J = 1,8
ISN 0375          405 CR(I,J) = CR(I,J) + U(K,I,J)*FACTOR
ISN 0376              CALL CMTMLT(CR,8,8,INT,8,32,SMOP)
ISN 0377              WRITE (6,1015) X
ISN 0378              DO 410 I=1,8
ISN 0379              DO 410 J=1,32
ISN 0380          410 OCOV(I,J) = SMOP(I,J)
ISN 0381              WRITE (6,1010) ((OCOV(I,J), J=1,32), I=1,8)
ISN 0382              IF (LENGTH .NE. X) GO TO 414
ISN 0384              DO 412 I=1,4
ISN 0385              DO 412 J=1,32
ISN 0386          412 SML(I,J) = OCOV(I+4,J)
ISN 0387          414 DO 415 I=1,8
ISN 0388              DO 415 J=1,32
ISN 0389          415 OCOV(I,J) = (0.0,-1.0)*SMOP(I,J)
ISN 0390              WRITE (6,1025)
ISN 0391              WRITE (6,1010) ((OCOV(I,J), J=1,32),I=1,8)
ISN 0392              GO TO 370
                C
                C     CALCULATION OF VARIANCES AND CORRELATIONS
                C
ISN 0393          450 DO 455 I=1,4
ISN 0394              DO 455 J=1,4
ISN 0395              VAR(I,J) = RSC(I,9*J - 8) + CONST(I,J)*(1.0-CONST(I,J))
ISN 0396          455 VAR(I,J+4) = SML(5-I,37 - 9*J) +  CONST(I,J+4)*(1.0-CONST(I,J+4))
ISN 0397              WRITE (6,925) ((VAR(I,J),J=1,8),I=1,4)
ISN 0398              DO 470 I=1,4
ISN 0399              DO 460 IALPHA =1,4
ISN 0400              IALP1 = IALPHA + 1
ISN 0401              DO 460 IBETA = IALP1,8
ISN 0402          460 CORR(IALPHA,IBETA) = (RSC(I,8*(IALPHA-1) + IBETA)  -
                     1 CONST(I,IALPHA)*CONST(I,IBETA))/DSQRT(VAR(I,IALPHA)*VAR(I,IBETA))
ISN 0403              DO 465 IALPHA = 5,7
ISN 0404              IALP1 = IALPHA + 1
ISN 0405              DO 465 IBETA = IALP1,8
ISN 0406          465 CORR(IALPHA,IBETA) = (SML(5-I,73 - 8*IALPHA - IBETA) -
                     1CONST(I,IALPHA)*CONST(I,IBETA))/DSQRT(VAR(I,IALPHA)*VAR(I,IBETA))
ISN 0407              DO 467 J=1,8
ISN 0408          467 CORR(J,J) = 1.0
ISN 0409              DO 468 I1=2,8
ISN 0410              I1M1 = I1 - 1
ISN 0411              DO 468 J1 = 1,I1M1
ISN 0412          468 CORR(I1,J1) = CORR(J1,I1)
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```
ISN 0413          WRITE (6,805) I
ISN 0414          WRITE (6,810) (VAR(I,J), J=1,8)
ISN 0415          DO 469 J1=1,8
ISN 0416      469 SDEV(J1) = DSQRT(VAR(I,J1))
ISN 0417          WRITE (6,820)
ISN 0418          WRITE (6,922) (CONST(I,J1), J1=1,8)
ISN 0419          WRITE (6,922) (SDEV(J1), J1=1,8)
ISN 0420          WRITE (6,815)
ISN 0421          WRITE (6,922) ((CORR(I1,J1),J1=1,8),I1=1,8)
ISN 0422          IF (I .NE. 1) GO TO 470
ISN 0424          DO 4690 I1 = 1,8
ISN 0425          DO 4690 J1 = 1,8
ISN 0426     4690 CORR(I1,J1) = CORR(I1,J1)*DSQRT(VAR(I,I1)*VAR(I,J1))
ISN 0427          WRITE (6,1030)
ISN 0428          WRITE (6,922) ((CORR(I1,J1), J1=1,8),I1=1,8)
ISN 0429          SUM1 = 0.0
ISN 0430          SUM2 = 0.0
ISN 0431          SUM3 = 0.0
ISN 0432          DO 4692 I1 = 1,4
ISN 0433          DO 4692 J1 = 1,4
ISN 0434          SUM1 = SUM1 + CORR(I1,J1)
ISN 0435          SUM2 = SUM2 + CORR(I1 + 4,J1 + 4)
ISN 0436     4692 SUM3 = SUM3 + CORR(I1,J1+4)
ISN 0437          SUM = SUM1 + SUM2 + 2.0*SUM3
ISN 0438          SR = 0.0
ISN 0439          ST = 0.0
ISN 0440          DO 4694 J1 = 1,4
ISN 0441          ST = ST + CONST(I,J1)
ISN 0442     4694 SR = SR + CONST(I,J1 + 4)
ISN 0443          SP = SR + ST
ISN 0444          SUM1 = DSQRT(SUM1)
ISN 0445          SUM2 = DSQRT(SUM2)
ISN 0446          SUM = DSQRT(SUM)
ISN 0447          WRITE (6,1040) LENGTH,M,STDEV
ISN 0448          WRITE (6,1035) I,SR,SUM2,ST,SUM1,SP,SUM
ISN 0449      470 CONTINUE
ISN 0450          IF (KCOUNT .EQ. NUMLTH) GO TO 130
ISN 0452          GO TO 190
ISN 0453      500 STOP
ISN 0454      805 FORMAT (////,25X,'VARIANCES AND CORRELATIONS FOR PARTICLES WITH IN
                 1ITIAL DIRECTION',I3)
ISN 0455      810 FORMAT (/,25X,'VARIANCES FOR PARTICLES EMERGING IN DIRECTIONS 1-4
                 1 AND 7-10',/,5X,8D14.7,/)
ISN 0456      815 FORMAT (/,25X,'CORRELATION MATRIX',/)
ISN 0457      820 FORMAT (26X,'MEANS AND STANDARD DEVIATIONS FOR PARTICLES WITH THIS
                 1 INITIAL DIRECTION,',/,26X,'EMERGENT DIRECTIONS ARE 1-4 AND 7-10,
                 2WITH MEANS DIRECTLY ABOVE THE STANDARD DEVIATIONS',/)
ISN 0458      835 FORMAT (1H0,40X,'RAYLEIGH SCATTERING',/)
ISN 0459      837 FORMAT (1H0,40X,'ISOTROPIC SCATTERING',/)
ISN 0460      840 FORMAT (1H0,5X,'MEAN NUMBER OF PARTICLES PRODUCED PER COLLISION IS
                 1',F10.5,'STANDARD DEVIATION IS',F10.5,/)
ISN 0461      880 FORMAT (10X,2F10.8,I3)
ISN 0462      900 FORMAT (10X,5F10.8)
ISN 0463      902 FORMAT (I3)
ISN 0464      905 FORMAT (1H1,40X,'MATRIX OF PROBABILITIES P(I,J)')
ISN 0465      908 FORMAT (40X,G16.4)
ISN 0466      910 FORMAT (10X,10F12.8)
ISN 0467      915 FORMAT (40X,'MATRIX R')
ISN 0468      918 FORMAT (40X,'LENGTH IS',F10.5)
ISN 0469      920 FORMAT (40X,'CHECK MATRIX')
ISN 0470      922 FORMAT (2X,8F16.7)
ISN 0471      925 FORMAT (10X,8D14.7)
ISN 0472      930 FORMAT (F8.4)
ISN 0473      935 FORMAT (1H1,10X,'MEAN MATRIX FOR TOTAL LENGTH =',F5.2,' WITH PARTI
                 1CLE AT POSITION X =',F5.2/)
ISN 0474      937 FORMAT (10X,'MEAN NUMBER OF PARTICLES PRODUCED PER COLLISSION IS',
                 1F8.4,/)
ISN 0475      940 FORMAT (10X,'ROW I REPRESENTS PARTICLE PRESENTLY TRAVELLING IN ITH
                 1 DIRECTION.  COLUMN J REPRESENTS PARTICLE EMERGING IN DIRECTION J'
                 2/)
ISN 0476      945 FORMAT (20X,'NO. OF ITERATIONS =',I3,'  TIME(MS) =',F8.0)
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ISN 0477          946 FORMAT (40X,'DETERMINANT IS',2D16.7,/)
ISN 0478          950 FORMAT (40X,'RETURNED MATRIX WITH EIGENVALUES'/)
ISN 0479          955 FORMAT (40X,'THETAS')
ISN 0480          960 FORMAT (40X,'EIGENVECTOR COLUMN MATRIX'/)
ISN 0481          964 FORMAT (2X,8D16.7/2X,8D16.7,/)
ISN 0482          965 FORMAT (2X,8D16.7,/)
ISN 0483          967 FORMAT (1H0,10X,'IMAGINARY PART OF MEAN MATRIX.  SHOULD BE ZERO'/)
ISN 0484          970 FORMAT (40X,'INVERSE MATRIX OF EIGENVECTORS'/)
ISN 0485          975 FORMAT (2X,8F16.7,/)
ISN 0486          985 FORMAT (40X,'EXP(R*LENGTH')
ISN 0487          986 FORMAT (40X,'EXP(R*LENGTH) USING H. G. SUBROUTINE'/)
ISN 0488          990 FORMAT (40X, 'CONSTANTS MATRIX'/)
ISN 0489         1005 FORMAT (1H1,5X,'CONSTANT MATRIX FOR SECOND FACTORIAL MOMENTS.  PAR
                     1TICLE AT 0,  I-TH ROW GIVES INITIAL DIRECTION 1-4, 7-10.',/5X,'(AL
                     2PHA,BETA)-TH MOMENT IN COL. 8(ALPHA-1) + BETA, ALPHA = 1 TO 4 AND
                     3BETA = 1 TO 4 AND (7 TO 10).')
ISN 0490         1010 FORMAT (4(2X,8D16.7/),/)
ISN 0491         1015 FORMAT (10X,'SECOND MOMENT OUTPUT, X = ',F8.4/)
ISN 0492         1020 FORMAT (1H0,20X,'IMAGINARY PART OF CONSTANTS MATRIX.  SHOULD BE ZE
                     1RO.',/)
ISN 0493         1025 FORMAT (//,20X,'IMAGINARY PART OF SECOND FACTORIAL MOMENT OUTPUT.
                     1 SHOULD BE ZERO.',//)
ISN 0494         1030 FORMAT (//,20X,'VARIANCE-COVARIANCE MATRIX FOR PARTICLE INCIDENT A
                     1T X = 0 TRAVELLING IN DIRECTION 1',//)
ISN 0495         1035 FORMAT (//,20X,'FOR PARTICLE AT X = 0 WITH INITIAL DIRECTION',I3,/
                     1,9X,'THE MEAN NUMBER OF PARTICLES REFLECTED IS',12X,F16.7,' WITH S
                     2TANDARD DEVIATION',F16.7,/,9X,'THE MEAN NUMBER OF PARTICLES TRANSM
                     3ITTED IS',10X,F16.7,' WITH STANDARD DEVIATION',F16.7,/,9X,'THE MEA
                     4N OF THE TOTAL NUMBER OF PARTICLES PRODUCED IS',F16.7,' WITH STAND
                     5ARD DEVIATION',F16.7,//)
ISN 0496         1040 FORMAT(/,20X,'THE LENGTH OF THE SLAB IS',F10.4,/,20X,'THE MEAN NUM
                     1BER OF PARTICLES PRODUCED PER COLLISION IS',F10.4,' WITH STANDARD
                     2DEVIATION',F10.4)
ISN 0497               END
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ACKNOWLEDGEMENT

We are indebted to J. E. Moyal and Harold Greenspan for a number of valuable discussions and suggestions in connection with this work.

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